Synthesizing Matrix Interpretations via Backward Completion*

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Abstract
Various approaches to automatically synthesizing termination proofs via matrix interpretations are used in state-of-the-art provers, most notably those based on satisfiability solving. In this talk, an alternative idea for the particular case of string rewriting is presented, exploiting the view of matrix interpretations as weighted automata. A demo of a prototype implementation will show the utility of this approach, as well as its limitations.

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1 Introduction

Matrix interpretations for string rewriting interpret the free monoid of strings in a ring structure, where concatenation of factors corresponds to multiplication and replacement of factors corresponds to subtraction. For termination, we use an (infinite) ordered ring, which is well-founded on its positive cone, see [1].

Example 1. In \((\mathbb{Z}, 0, 1, +, \cdot)\) we can prove termination of \(\{aba \rightarrow aa\}\) by the interpretation \(i : a \mapsto 1, b \mapsto 2\) as \(i(aba) - i(aa) = i(a) \cdot i(b) \cdot i(a) = 1 > 0\), but neither \(\{ab \rightarrow ba\}\) nor \(\{aa \rightarrow aba\}\) can be proven terminating in \(\mathbb{Z}\) as multiplication is commutative and due to the totality of the ordering employed, respectively.

For that reason, we use non-commutative rings with a non-total ordering, in this case rings of square matrices over the natural numbers; for definitions and notations we refer to [1].

Example 2. For proving termination of \(\{ab \rightarrow ba\}\) consider the \(E_2\)-interpretation \(i : a \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), where \(i(ab) - i(ba) = i(a) \cdot i(b) - i(b) \cdot i(a) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in P_2\), and for proving termination of \(\{aa \rightarrow aba\}\) the \(E_1\)-interpretation \(i : a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\) gives \(i(aa) - i(aba) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in P_1\).

Various approaches for synthesizing matrix interpretations have been proposed, for instance complete enumeration of restricted matrix shapes, random guesses for small matrix dimensions, evolutionary programming, and, most prominently, constraint solving. In this talk, we present backward completion as another approach for the same purpose. This is related to forward completion procedures for match-bound termination proofs as in [2].

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2 Matrix interpretations as weighted automata

Matrix interpretations correspond to weighted automata in a direct way. A weighted automaton is a mapping \( \text{weight} : Q \times \Sigma \times Q \to \mathbb{N} \), where \( Q \) is a finite set (of states). This mapping is extended to weight \( \text{weight} : Q \times \Sigma^* \times Q \to \mathbb{N} \) by multiplying weights along a single path and adding weights of different paths. A transition from state \( p \) to state \( q \) with weight \( n \) for letter \( a \) corresponds to \( i(a)_{p,q} = n \) in a matrix interpretation, and vice versa.

▶ Example 3. The matrix interpretation
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

of type \( E_{\{1,5\}} \) corresponds to the following weighted automaton, proving termination of \( \{aabb \to bbbaa\} \) (TPDB problem z001, see [3], known as Zantema’s problem):

3 Backward completion

Often, automata of a particularly simple form can prove termination of string rewriting systems. These straight-line automata essentially consist of a single path that corresponds to the left-hand side of a rule, where each transition has weight 1.

▶ Example 4. For \( \{aa \to aba\} \), the following automaton proves termination:

▶ Example 5. In the same way, for \( \{bbcabc \to abcbca\} \) (TPDB problem z061) the automaton

serves as a termination certificate.

Of course, straight-line automata fail in many cases.

▶ Example 6. Consider \( \{bcabbc \to abcbbca\} \) (TPDB problem z062). For the automaton

we get weight(1, bcabbc, 4) = 0 \neq 1 = weight(1, abcbbca, 4), and the termination proof fails. However, if we compensate for that defect by adding an edge

we are done: weight(1, bcabbc, 4) = 1 = weight(1, abcbbca, 4).
This motivates backward completion. Define a pair of states \((s, t)\) with weight\((s, \ell, t) <\) weight\((s, r, t)\) to be a defect for the rewriting rule \(\ell \rightarrow r\). Compensate for such a defect by adding a path for \(\ell\) from \(s\) to \(t\). In general, for a backward completion step choose a defect \((s, t)\) for some rule \(\ell \rightarrow r\) and a factorization \(\ell = \ell_1\ell_2\ell_3\) so that, for some states \(p, q,\)

\[
\begin{array}{c}
s \xrightarrow{\ell_1} p \\
\uparrow
\end{array}, \quad \begin{array}{c}
q \xrightarrow{\ell_3} t \\
\uparrow
\end{array}
\]

and add a path of weight 1 from \(p\) to \(q\) for \(\ell_2:\)

\[
\begin{array}{c}
s \xrightarrow{\ell_1} p \xrightarrow{\ell_2} q \\
\uparrow
\end{array}, \quad \begin{array}{c}
q \xrightarrow{\ell_3} t \\
\uparrow
\end{array}
\]

4 Implementation

One possibility is to implement backward completion as a probabilistic algorithm, as follows.

1. Choose as start automaton \(\Sigma: \begin{array}{c}
0 \\
\uparrow
\end{array} \xrightarrow{\ell_1} \begin{array}{c}
1 \\
\uparrow
\end{array} \Sigma: 1\) with just two states.
2. Randomly choose the left-hand side \(\ell\) of a rule \(\ell \rightarrow r\) and add a path from \(0\) to \(1\) for \(\ell\), resulting in \(\Sigma: \begin{array}{c}
0 \\
\uparrow
\end{array} \xrightarrow{\ell_1} \begin{array}{c}
1 \\
\uparrow
\end{array} \Sigma: 1\) . Note that now weight\((0, \ell, 1) >\) weight\((0, r, 1)\) unless \(\ell\) is a factor of \(r\), in which case the rewriting system is already non-terminating.
3. As long as defects exist and weight\((0, \ell, 1) >\) weight\((0, r, 1)\) keeps holding true, randomly perform some backward completion step.

This might fail or go on forever, but in case the procedure stops successfully, the resulting automaton provides a termination certificate.

▶ Example 7. The automaton in example 3 can easily be synthesized in this way. Further examples include rewriting systems from the TPDB that haven’t been solved during the 2013 termination competition (see termination-portal.org/wiki/Termination_Competition/).

5 Extensions

In the talk, we present various variants of this strategy, in particular a non-probabilistic variant where completion steps are enumerated under a breadth-first strategy. Building on results from [4], a specialized strategy aims at automatically proving polynomial upper bounds on derivation lengths. Backward completion can be easily adapted to the setting of relative termination proofs, thereby considerably strengthening its applicability. A demo implementation shows the utility of the approach, as well as its limitations.

References

3 The Termination Problems Data Base, termination-portal.org/wiki/TPDB/.