Predicative Lexicographic Path Orders: Towards a Maximal Model for Primitive Recursive Functions*

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Abstract

The predicative lexicographic path order (PLPO for short), a syntactic restriction of the lexicographic path order, is presented. As well as lexicographic path orders, several non-trivial primitive recursive equations, e.g., primitive recursion with parameter substitution, unnested multiple recursion, or simple nested recursion, can be oriented with PLPOs. It can be shown that PLPOs however only induce primitive recursive upper bounds for derivation lengths of compatible rewrite systems. This yields an alternative proof of a classical fact that the class of primitive recursive functions is closed under these non-trivial primitive recursive equations.

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1 Introduction

As shown by R. Péter [9], the class of primitive recursive functions is closed under a recursion schema that is not an instance of primitive recursion, e.g., primitive recursion with parameter substitution (**PRP**) $f(x+1,y) = h(x,y,f(x,p(x,y)))$, unnested multiple recursion (**UMR**) $f(x+1,y+1) = h(x,y,f(x,p(x,y)),f(x+1,y))$, or simple nested recursion (**SNR**) $f(x+1,y) = h(x,y,f(x,p(x,y,f(x,y))))$. H. Simmons [10] showed Péter's results in a general framework aiming to answer a deep question why primitive recursive functions are closed under these non-trivial primitive recursive equations. As observed by E. A. Cichon and A. Weiermann [6], in order to assess the complexity of a given function, we can discuss about maximal lengths of rewriting sequences, which is known as derivation lengths, in a term rewrite system that defines the function. More precisely, if every derivation length in a given rewrite system $\mathcal{R}$ is bonded by a function in a class $\mathcal{F}$, then the function defined by $\mathcal{R}$ is elementary recursive in $\mathcal{F}$ measured by the size of a starting term. In [2] M. Avanzini and G. Moser have shown that "elementary recursive in" can be replaced by "polynomial time in" if one only considers of rewriting sequences starting with terms whose arguments are already normalised. In [6] alternative proofs of Péter's results were given employing primitive recursive number-theoretic interpretations of rewrite systems corresponding to those non-trivial primitive recursive equations mentioned above. On the other side, any equation of (**PRP**), (**UMR**) and (**SNR**) can be oriented with a termination order known as the lexicographic path order (LPO for short). As shown by Weiermann [11], LPOs induce multiply recursive upper bounds for those derivation lengths. Thus, in order to discuss about (**PRP**), (**UMR**) or (**SNR**), it is natural to restrict LPOs. In [5] Cichon

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introduced the ramified lexicographic path order (RLPO for short), a syntactic restriction of LPO, capturing (**PRP**) and (**UMR**). This work is an attempt to find a *maximal* model for primitive recursive functions based on termination orders in a way different from [5] but stemming from Simmons' approach in [10]. The recursion-theoretic characterisation given in [10] is based on a restrictive (higher order primitive) recursion that is commonly known as *predicative recursion*. A brief explanation about predicative recursion can be found in the paragraph after Example 5 on page 3. Taking the idea of predicative recursion into the lexicographic comparison, we introduce the *predicative lexicographic path order* (PLPO for short), a syntactic restriction of LPO. As well as LPOs, (**PRP**) (**UMR**) and (**SNR**) can be oriented with PLPOs. However, in contrast to LPOs, PLPOs only induce primitive recursive upper bounds for derivation lengths of compatible rewrite systems. This yields an alternative proof of the fact that primitive recursive functions are closed under (**PRP**) (**UMR**) and (**SNR**). The definition of PLPO is also strongly motivated by a more recent work [1] by Avanzini, Moser and the author.

2 Predicative Lexicographic Path Orders

Let $\mathcal{V}$ denote a countably infinite set of variables. A *signature* $\mathcal{F}$ is a finite set of function symbols. The number of argument positions of a function symbol $f \in \mathcal{F}$ is denoted as $\mathsf{arity}(f)$. We write $\mathcal{T}(\mathcal{V}, \mathcal{F})$ to denote the set of terms over $\mathcal{V}$ and $\mathcal{F}$. The signature $\mathcal{F}$ can be partitioned into the set $\mathcal{C}$ of *constructors* and the set $\mathcal{D}$ of *defined* symbols. We suppose that $\mathcal{C}$ contains at least one constant. We assume a specific (possibly empty) subset $\mathcal{D}_{\mathsf{lex}}$ of $\mathcal{D}$. A *precedence* $\geqslant_{\mathcal{F}}$ on the signature $\mathcal{F}$ is a quasi-order whose strict part $>_{\mathcal{F}}$ is well-founded on $\mathcal{F}$. We write $f \approx_{\mathcal{F}} g$ if $f \geqslant_{\mathcal{F}} g$ and $g \geqslant_{\mathcal{F}} f$. We also assume that the argument positions of every function symbol are separated into two kinds. The separation is indicated by semicolon as $f(t_1, \dots, t_k; t_{k+1}, \dots, t_{k+l})$, where $t_1, \dots, t_k$ are called *normal* arguments whereas $t_{k+1}, \dots, t_{k+l}$ are called *safe* ones. The equivalence $\approx_{\mathcal{F}}$ is extended to the term equivalence $\approx$. We write $f(s_1, \dots, s_k; s_{k+1}, \dots, s_{k+l}) \approx g(t_1, \dots, t_k; s_{k+1}, \dots, t_{k+l})$ if $f \approx_{\mathcal{F}} g$ and $s_j \approx t_j$ for all $j \in \{1, \dots, k+l\}$. An auxiliary relation $s = f(s_1, \dots, s_k; s_{k+1}, \dots, s_{k+l}) \sqsupset_{\mathsf{plpo}} t$ holds if one of the following cases holds, where $s \sqsupseteq_{\mathsf{plpo}} t$ denotes $s \sqsupset_{\mathsf{plpo}} t$ or $s \approx t$.

1. $f \in \mathcal{C}$ and $s_i \sqsupseteq_{\mathsf{plpo}} t$ for some $i \in \{1, \dots, k+l\}$.
2. $f \in \mathcal{D}$ and $s_i \sqsupseteq_{\mathsf{plpo}} t$ for some $i \in \{1, \dots, k\}$.
3. $f \in \mathcal{D}$, $t = g(t_1, \dots, t_m; t_{m+1}, \dots, t_{m+n})$ for some g such that $f >_{\mathcal{F}} g$, and $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{1, \dots, m+n\}$.

Now we define the *predicative lexicographic path order* (PLPO for short) denoted as $>_{\mathsf{plpo}}$. We write $s \geqslant_{\mathsf{plpo}} t$ if $s >_{\mathsf{plpo}} t$ or $s \approx t$, like the relation $\sqsupseteq_{\mathsf{plpo}}$, write $(s_1, \dots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \dots, t_k)$ if $s_j \geqslant_{\mathsf{plpo}} t_j$ for all $j \in \{1, \dots, k\}$, and we write $(s_1, \dots, s_k) >_{\mathsf{plpo}} (t_1, \dots, t_k)$ if $(s_1, \dots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \dots, t_k)$ and additionally $s_i >_{\mathsf{plpo}} t_i$ for some $i \in \{1, \dots, k\}$.

▶ **Definition 1.** $s = f(s_1, \dots, s_k; s_{k+1}, \dots, s_{k+l}) >_{\mathsf{plpo}} t$ holds if one of the following holds.

1. $s \sqsupset_{\mathsf{plpo}} t$.
2. $s_i \geqslant_{\mathsf{plpo}} t$ for some $i \in \{1, \dots, k+l\}$.
3. $f \in \mathcal{D}$, $t = g(t_1, \dots, t_m; t_{m+1}, \dots, t_{m+l})$ for some g such that $f >_{\mathcal{F}} g$, $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{1, \dots, m\}$, and $s >_{\mathsf{plpo}} t_j$ for all $j \in \{m+1, \dots, m+n\}$.
4. $f \in \mathcal{D} \backslash \mathcal{D}_{\mathsf{lex}}$, $t = g(t_1, \dots, t_k; t_{k+1}, \dots, t_{k+l})$ for some g such that $f \approx_{\mathcal{F}} g$, $(s_1, \dots, s_k) \geqslant_{\mathsf{plpo}} (t_1, \dots, t_k)$, and $(s_{k+1}, \dots, s_{k+l}) >_{\mathsf{plpo}} (t_{k+1}, \dots, t_{k+l})$.
5. $f \in \mathcal{D}_{\mathsf{lex}}$, $t = g(t_1, \dots, t_m; t_{m+1}, \dots, t_{m+l})$ for some g such that $f \approx_{\mathcal{F}} g$, and there exists $i_0 \in \{1, \dots, \min(k, m)\}$ such that $s_j \approx t_j$ for all $j \in \{1, \dots, i_0 - 1\}$, $s_{i_0} >_{\mathsf{plpo}} t_{i_0}$, $s \sqsupset_{\mathsf{plpo}} t_j$ for all $j \in \{i_0 + 1, \dots, m\}$, and $s >_{\mathsf{plpo}} t_j$ for all $j \in \{m+1, \dots, m+n\}$.

By induction according to the definition of $>_{\mathsf{plpo}}$, the inclusion $>_{\mathsf{plpo}} \subseteq >_{\mathsf{lpo}}$ can be shown for the LPO $>_{\mathsf{lpo}}$ induced by the same precedence. The converse inclusion does not hold in general.

▶ **Example 2.** $\mathcal{R}_{\mathsf{PR}} = \{\mathsf{f}(;0,y) \to \mathsf{g}(;y), \mathsf{f}(;\mathsf{s}(;x),y) \to \mathsf{h}(;x,y,\mathsf{f}(;x,y))\}$.

The sets $\mathcal{C}$ and $\mathcal{D}$ are defined by $\mathcal{C} = \{0, \mathsf{s}\}$ and $\mathcal{D} = \{\mathsf{g}, \mathsf{h}, \mathsf{f}\}$. Let $\mathcal{D}_{\mathsf{lex}} = \emptyset$. Define a precedence $\geqslant_{\mathcal{F}}$ by $\mathsf{f} \approx_{\mathcal{F}} \mathsf{f}$ and $\mathsf{f} >_{\mathcal{F}} \mathsf{g}, \mathsf{h}$. Define an argument separation as indicated in the rules. Then $\mathcal{R}_{\mathsf{PR}}$ can be oriented with the PLPO $>_{\mathsf{plpo}}$ induced by $\geqslant_{\mathcal{F}}$. For the first rule $\mathsf{f}(;0,y) >_{\mathsf{plpo}} y$ and hence $\mathsf{f}(;0,y) >_{\mathsf{plpo}} \mathsf{g}(;y)$ by Case 3 in Definition 1. Consider the second rule. Since $(\mathsf{s}(;x),y) >_{\mathsf{plpo}} (x,y)$, $\mathsf{f}(;\mathsf{s}(;x),y) >_{\mathsf{plpo}} \mathsf{f}(;x,y)$ holds as an instance of Case 4. An application of Case 3 allows us to conclude $\mathsf{f}(;\mathsf{s}(;x),y) >_{\mathsf{plpo}} \mathsf{h}(;x,y,\mathsf{f}(;x,y))$.

▶ **Example 3.** $\mathcal{R}_{\mathsf{PRP}} = \{\mathsf{f}(0;y) \to \mathsf{g}(;y), \mathsf{f}(s(;x);y) \to \mathsf{h}(x;y,\mathsf{f}(x;\mathsf{p}(x;y)))\}$.

The sets $\mathcal{C}$ and $\mathcal{D}$ are defined as in the previous example. Define the set $\mathcal{D}_{\mathsf{lex}}$ by $\mathcal{D}_{\mathsf{lex}} = \{\mathsf{f}\}$. Define a precedence $\geqslant_{\mathcal{F}}$ by $\mathsf{f} \approx_{\mathcal{F}} \mathsf{f}$ and $\mathsf{f} >_{\mathcal{F}} \mathsf{q}$ for all $\mathsf{q} \in \{\mathsf{g}, \mathsf{p}, \mathsf{h}\}$. Define an argument separation as indicated. Then $\mathcal{R}_{\mathsf{PRP}}$ can be oriented with the induced PLPO $>_{\mathsf{plpo}}$. We only consider the most interesting case. Namely we oriente the second rule. Since $\mathsf{s}(;x) \sqsupset_{\mathsf{plpo}} x$, $\mathsf{f}(\mathsf{s}(;x);y) \sqsupset_{\mathsf{plpo}} x$ holds by the definition of $\sqsupset_{\mathsf{plpo}}$. This together with Case 3 yields $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{p}(x;y)$. Hence an application of Case 5 yields $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{f}(x;\mathsf{p}(x;y))$. Another application of Case 3 allows us to conclude $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{h}(x;y,\mathsf{f}(x;\mathsf{p}(x;y)))$.

▶ **Example 4.** $\mathcal{R}_{\mathsf{UMR}} = \left\{ \begin{array}{rcl} \mathsf{f}(0,y;) & \to & \mathsf{g}_0(y;), \\ \mathsf{f}(\mathsf{s}(;x),0;) & \to & \mathsf{g}_1(x;\mathsf{f}(x,\mathsf{q}(x;);)), \\ \mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;y);) & \to & \mathsf{h}(x,y;\mathsf{f}(x,\mathsf{p}(x,y;);),\mathsf{f}(\mathsf{s}(;x),y;)) \end{array} \right\}$.

The sets $\mathcal{C}$ and $\mathcal{D}$ are defined as in the former two examples and the set $\mathcal{D}_{\mathsf{lex}}$ is defined in the previous example. Define a precedence $\geqslant_{\mathcal{F}}$ by $\mathsf{f} \approx_{\mathcal{F}} \mathsf{f}$ and $\mathsf{f} >_{\mathcal{F}} \mathsf{g}$ for all $\mathsf{g} \in \{\mathsf{g}_0, \mathsf{g}_1, \mathsf{p}, \mathsf{q}, \mathsf{h}\}$. Define an argument separation as indicated. Then $\mathcal{R}_{\mathsf{UMR}}$ can be oriented with the induced PLPO $>_{\mathsf{plpo}}$. Let us consider the most interesting case. Namely we oriente the third rule. Since $\mathsf{f} >_{\mathcal{F}} \mathsf{p}$ and $\mathsf{s}(;u) \sqsupset_{\mathsf{plpo}} u$ for each $u \in \{x, y\}$, $\mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;y);) \sqsupset_{\mathsf{plpo}} \mathsf{p}(x,y;)$ holds by the definition of $\sqsupset_{\mathsf{plpo}}$. Hence, since $\mathsf{s}(;x) >_{\mathsf{plpo}} x$, an application of Case 5 in Definition 1 yields $\mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;x);) >_{\mathsf{plpo}} \mathsf{f}(x,\mathsf{p}(x,y;);)$. Another application of Case 5 yields $\mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;y);) >_{\mathsf{plpo}} \mathsf{f}(\mathsf{s}(;x),y;)$. Clearly $\mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;y);) \sqsupset_{\mathsf{plpo}} u$ for each $u \in \{x, y\}$. Hence an application of Case 3 allows us to conclude $\mathsf{f}(\mathsf{s}(;x),\mathsf{s}(;y);) >_{\mathsf{plpo}} \mathsf{h}(x,y;\mathsf{f}(x,\mathsf{p}(x,y;);),$ $\mathsf{f}(\mathsf{s}(;x),y;))$.

▶ **Example 5.** $\mathcal{R}_{\mathsf{SNR}} = \{\mathsf{f}(0;y) \to \mathsf{g}(;y), \mathsf{f}(\mathsf{s}(;x);y) \to \mathsf{h}(x;y,\mathsf{f}(x;\mathsf{p}(x;y,\mathsf{f}(x;y))))\}$.

The sets $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{D}_{\mathsf{lex}}$ are defined as in the former three examples. Define a precedence $\geqslant_{\mathcal{F}}$ as in the previous example. Define an argument separation as indicated. Then $\mathcal{R}_{\mathsf{SNR}}$ can be oriented with the induced PLPO $>_{\mathsf{plpo}}$. We only oriente the second rule. As we observed in the previous example, $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{f}(x;y)$ holds by Case 5. Hence $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{p}(x;y,\mathsf{f}(x;y))$ holds by Case 3. This together with Case 5 yields $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{f}(x;\mathsf{p}(x;y,\mathsf{f}(x;y)))$. Thus another application of Case 3 allows us to conclude $\mathsf{f}(\mathsf{s}(;x);y) >_{\mathsf{plpo}} \mathsf{h}(x;y,\mathsf{f}(x;\mathsf{p}(x;y,\mathsf{f}(x;y))))$.

Careful readers may observe that the general form of nested recursion, e.g., defining equations for the Ackermann function, cannot be oriented with PLPOs. As intended in [4], predicative recursion is a syntactic restriction of the standard (primitive) recursion, where the number of recursive calls is measured only by a normal argument whereas results of recursion are allowed to be substituted only for safe arguments: $f(x+1,\vec{y};\vec{z}) = h(x,\vec{y};\vec{z},f(x,\vec{y};\vec{z}))$. In [10] the meaning of predicative recursion is modified (though [10] is an earlier work than [4]) in such a way that recursive calls are allowed even on safe arguments for the standard primitive recursion (see Example 2) but still restricted on normal arguments for multiple (nested) recursion (see Example 3–5). In the sequel we present a primitive recursive interpretation for PLPOs. This yields that the maximal length of rewriting sequences in any rewrite system compatible with a PLPO is bounded by a primitive recursive function in the size of the starting term. All the missing details can be found in a technical report [7]. Following [6, page 214], given a natural $d \geq 2$, we define the primitive recursive function F_m by $F_0(x) = d^{x+1}$ and $F_{m+1}(x) = F_m^{d(1+x)}(x)$, where F_m^d denotes the d-fold iteration of F_m.

▶ **Definition 6.** Given k, we inductively define the k-ary primitive recursive function $F_{m,n}$ by $F_{m,0}(x_1,\dots,x_k) = 0$, $F_{m,n+1}(x_1,\dots,x_k) = \begin{cases} F_m^{F_{m,n}(x_1,\dots,x_k)+d(1+x_{n+1})}(\sum_{j=1}^{n+1} x_j) & \text{if } n < k, \\ F_m^{F_{m,n}(x_1,\dots,x_k)}(\sum_{j=1}^{k} x_j) & \text{if } k \leq n. \end{cases}$

▶ **Definition 7.** Let ℓ be a natural such that $2 \leq \ell$, $\mathcal{F}$ a signature and $\geqslant_{\mathcal{F}}$ a precedence on $\mathcal{F}$. The *rank* $\mathsf{rk} : \mathcal{F} \to \mathbb{N}$ is defined in accordance with $\geqslant_{\mathcal{F}}$, i.e., $\mathsf{rk}(f) \geq \mathsf{rk}(g) \Leftrightarrow f \geqslant_{\mathcal{F}} g$. Define a natural K by $K = \max\{k \mid f \in \mathcal{F} \text{ and } f \text{ has } k \text{ normal argument positions}\}$. Then a primitive recursive interpretation $\mathcal{I} : \mathcal{T}(\mathcal{F}) \to \mathbb{N}$ is defined by $\mathcal{I}(t) = d^{F_{\mathsf{rk}(f)+\ell,K+1}(\mathcal{I}(t_1),\dots,\mathcal{I}(t_k))} \cdot \sum_{j=1}^{l} \mathcal{I}(t_{k+j})$, where $t = f(t_1,\dots,t_k;t_{k+1},\dots,t_{k+l}) \in \mathcal{T}(\mathcal{F})$.

Let $2 \leq \ell$. We define a restriction $\sqsupset^{\ell}_{\mathsf{plpo}}$ of $\sqsupset_{\mathsf{plpo}}$: $s = f(s_1,\dots,s_k;s_{k+1},\dots,s_{k+l}) \sqsupset^{\ell}_{\mathsf{plpo}} t$ holds if one of the following cases holds, where $s \sqsupseteq^{\ell}_{\mathsf{plpo}} t$ denotes $s \sqsupset^{\ell}_{\mathsf{plpo}} t$ or $s \approx t$.

1. $f \in \mathcal{C}$ and $s_i \sqsupseteq^{\ell}_{\mathsf{plpo}} t$ for some $i \in \{1,\dots,k+l\}$.
2. $f \in \mathcal{D}$ and $s_i \sqsupseteq^{\ell}_{\mathsf{plpo}} t$ for some $i \in \{1,\dots,k\}$.
3. $f \in \mathcal{D}$, $t = g(t_1,\dots,t_m;t_{m+1},\dots,t_{m+n})$ for some g such that $f >_{\mathcal{F}} g$, and $s \sqsupset^{\ell-1}_{\mathsf{plpo}} t_j$ for all $j \in \{1,\dots,m+n\}$.

We write $>^{\ell}_{\mathsf{plpo}}$ to denote the PLPO induced by $\sqsupset^{\ell}_{\mathsf{plpo}}$ and $|t|$ to denote the size of a term t. In addition, for a rewrite system $\mathcal{R}$ and a relation $>$, we write $\mathcal{R} \subseteq >$ if $l > r$ holds for every rule $(l \to r) \in \mathcal{R}$.

▶ **Lemma 8.** *Let $s,t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ and $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ be a ground substitution. Suppose* $\max(\{\mathsf{arity}(f) \mid f \in \mathcal{F}\} \cup \{\ell \cdot (K+2)+2\} \cup \{|t|+1\}) \leq d$. *If $s >^{\ell}_{\mathsf{plpo}} t$, then, for the interpretation $\mathcal{I}$ induced by ℓ and d, $\mathcal{I}(s\sigma) > \mathcal{I}(t\sigma)$ holds.*

▶ **Lemma 9.** *Let $s,t \in \mathcal{T}(\mathcal{F})$ be ground terms and $C(\Box) \in \mathcal{T}(\mathcal{F} \cup \{\Box\})$ a (ground) context. If $\mathcal{I}(s) > \mathcal{I}(t)$, then $\mathcal{I}(C(s)) > \mathcal{I}(C(t))$ holds.*

▶ **Theorem 10.** *Let $\mathcal{R}$ be a rewrite system over a signature $\mathcal{F}$ such that $\mathcal{R} \subseteq >^{\ell}_{\mathsf{plpo}}$ for some $\ell \geq 2$ and $s,t \in \mathcal{T}(\mathcal{F})$ be ground terms. Suppose* $\max(\{\mathsf{arity}(f) \mid f \in \mathcal{F}\} \cup \{\ell \cdot (K+2)+2\} \cup \{|r|+1 \mid (l \to r) \in \mathcal{R}\}) \leq d$. *If $s \to_{\mathcal{R}} t$, then, for the interpretation induced by ℓ and d, $\mathcal{I}(s) > \mathcal{I}(t)$ holds.*

▶ **Theorem 11.** *For any rewrite system $\mathcal{R}$ such that $\mathcal{R} \subseteq >_{\mathsf{plpo}}$ for some PLPO $>_{\mathsf{plpo}}$, the length of any rewriting sequence in $\mathcal{R}$ starting with a ground term is bounded by a primitive recursive function in the size of the starting term.*

▶ **Corollary 12.** *The class of primitive recursive functions is closed under primitive recursion with parameter substitution, unnested multiple recursion and simple nested recursion.*

3 Concluding remarks

A novel termination order, the predicative lexicographic path order PLPO, was presented. As well as LPOs, any instance of (**PRP**), (**UMR**) and (**SNR**) can be oriented with a PLPO. Note that *general* simple nested recursion briefly discussed in [6, page 221], e.g., simple nested recursion with more than one recursion parameters, can be even oriented with PLPOs. On the other side, PLPOs only induce primitive recursive upper bounds for derivation lengths of compatible rewrite systems. It turns out that the presented primitive recursive interpretation is not affected even if in Case 4 of Definition 1 one allows permutations of safe argument positions on $\{k+1, \ldots, k+l\}$. Allowance of permutations of normal argument positions is not clear at present. One would recall that, as shown by D. Hofbauer in [8], *multiset path orders* only induce primitive recursive upper bounds for derivation lengths of compatible rewrite systems. Allowance of multiset comparison is not clear in the case even for safe arguments. We mention that every PLPO is a slight extension of an *exponential path order* EPO* defined in [1] though EPO*s only induce exponential (innermost) derivational complexity. An auxiliary relation $\sqsupset_{\mathsf{epo}}$ employed to define EPO* is strictly included in $\sqsupset_{\mathsf{plpo}}$. We also mention that the auxiliary relation $\sqsupset_{\mathsf{plpo}}$ is exactly the same as the relation $>_{\mathsf{pop}}$ introduced in [3] to define the *polynomial path order* POP*. By induction according to the inductive definition of an EPO* $>_{\mathsf{epo*}}$, it can be shown that $>_{\mathsf{epo*}} \subseteq >_{\mathsf{plpo}}$ holds with the same precedence and the same argument separation. In general none of (**PRP**), (**UMR**) and (**SNR**) can be oriented with EPO*s. Perhaps it should be emphasised that a significant difference between PLPO and EPO* lies in Case 4 of Definition 1. Without Case 4 PLPOs would only induce elementary recursive derivational complexity.

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