SAT compilation for Termination Proofs via Semantic Labelling

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Abstract

For the termination method of rule removal by semantic labelling, matrix interpretations and then unlabelling, we give a purely functional specification and apply our CO4 constraint compiler to automatically generate a propositional encoding. This extends a “manual” SAT encoding of this method in Jambox (Endrullis, 2008): we allow sequences of interpretations during labelling.

1 Introduction

Finding parameters for termination proof methods is a constraint satisfaction problem: given a rewrite system, we are looking for a precedence for a path order, or for coefficients of interpretations, etc.

For (finite domain) constraint programming, SAT (propositional satisfiability) takes the role of an assembly language: it gives direct access to the machine (i.e., powerful and highly optimized SAT solver), but it is cumbersome and error-prone for actual programming. Instead, we want to use a high-level specification language, to increase expressiveness and safety.

Two of us (Bau, Waldmann) are developing the CO4 language and compiler \cite{CO4} that translates Haskell specifications to SAT formulas. We have previously applied CO4 for finding loops in string rewriting. With respect to a manual SAT encoding of TTT2 \cite{TTT2}, our compiled formula is larger by a factor of $< 10$, with a similar factor for run-times of the SAT solver, but the source code of the constraint system is just 1/3 in size.

In this note, we report on an application of CO4 in termination of string rewriting, using the method of semantic labelling, interpretations, and unlabelling. This is well-known \cite{SemanticLabelling}, and it had been implemented in several termination tools already: Teparla and Torpa used semantic labelling in $\approx$ 2006, by a stochastic search for models.

Jambox \cite{Jambox} $\approx$ 2007 used a “manual” SAT encoding for the following constraint: given a rewrite system $R$, there is a model $M$ for $R$ and an interpretation $I$ for the $M$-labelled system $R_M$ and a non-empty $S \subseteq R$ such that each rule in $S_M$ is strictly $I$-decreasing. We then have relative termination $SN(S/R)$, that is, we can remove $S$. Correctness of this method had been formalized for CeTA \cite{CeTA}.

We now extend Jambox' implementation by allowing for a sequence of interpretations for the labelled system. This is a conceptually simple modification: instead of one order, use a lexicographic product of several orders. Using the CO4 language, this modification can be expressed directly in the source code (see function \texttt{lexi}), and all extra encoding is done by the compiler.

2 Semantic Labelling and Unlabelling

An algebra $A$ is a model for a rewrite system $R$ over signature $\Sigma$ if for each $s \rightarrow_R t$, we have $A(s) = A(t)$. If we have a model $A$ for $R$, we construct a labelled version $R_A$ of $R$ where each function symbol is labelled by the value(s) of its argument(s).
Example 1. We consider the string rewriting system $R = \{aa \rightarrow aba\}$ over the signature $\Sigma = \{a, b\}$. The algebra $A$ with domain $D = \{0, 1\}$, and interpretation $a_A = \{(0, 1), (1, 1)\}, b_A = \{(0, 0), (1, 0)\}$, is a model for $R$, and $R_A$ is $\{a_1a_0 \rightarrow a_0b_1a_0, a_1a_1 \rightarrow a_0b_1a_1\}$.

Termination of $R$ and $R_A$ are equivalent (under some conditions that are true for string rewriting). The point of the method is that termination of $R_A$ may be easier to show that termination of $R$ since we increase the signature, and have more room to maneuver.

In the example, we can see that the number of occurrences of $a_1$ decreases in each rule application. This can be verified by the (linear, additive) interpretation $\{(a_0, 0), (a_1, 1), (b_1, 0)\}$ for $R_A$. There is no such interpretation for $R$.

3 SAT compilation with CO4

CO4 [1] is a high-level declarative language for describing constraint systems. The language is a subset of the purely functional programming language Haskell [3] that includes user-defined algebraic data types and recursive functions defined by pattern matching, as well as higher-order polymorphic types.

Source programs operate on algebraic data (like Booleans, List, Trees) which we call concrete values. CO4 compilation creates programs that operate on abstract values. An abstract value $\in A$ represents a set of concrete values of the same type. An abstract value is a tree, where each node contains a sequence of propositional logic formulas. Given a truth assignment $\sigma$, the sequence of truth values of these formulas under $\sigma$ gives a binary number that denotes a constructor. Doing this for each node, a concrete value $\text{decode}(a, \sigma)$ is determined.

A high level, parametric constraint system $\text{constraint :: } P \rightarrow U \rightarrow \text{Bool}$ written in CO4 represents a predicate on $U$ depending on a parameter $p \in P$. $p$ is not known a-priori. The compilation and evaluation of a CO4 program is a staged process:

1. The original program, operating on concrete values $C$, is compiled into an abstract program that operates on abstract values $A$.
2. Given an parameter $p$, the abstract program is evaluated, resulting in an abstract value $a$ (representing a Boolean).
3. The formula $a = \text{True}$ is given to an external SAT solver. It tries to find a satisfying assignment $\sigma$.
4. A value $u \in U$ is reconstructed by $\sigma$, so that $\text{constraint } p \ u = \text{True}$.

Values that depend on the $u$ parameter of the top-level constraint are not known during abstract evaluation. Abstract evaluation of case-distinctions on those values has to evaluate (abstractly) all branches and then merge the results into a single abstract value.

Natural numbers may be defined as list of Booleans $\text{type Nat = [Bool]}$ in CO4. For naturals with bit-width $k$ this encoding ends up in abstract values with depth $k$. A user-defined function on naturals would require $k$ pattern matches to inspect a number: this leads to long runtimes of the compiled program and the resulting formulas would be large. To avoid those issues, CO4 provides built-in naturals, where a number of bit-width $k$ is encoded by an abstract value with $k$ flags and no arguments. CO4 provides arithmetic and relational operations on naturals as well.
4 Implementation

Complete source code of the termination method is available as part of https://github.com/jwaldmann/matchbox (file MB/Label/SLPO.standalone.hs).

Here, we just indicate the main types and functions. For string rewriting, we have

```haskell
type Symbol = [Bool]; type Word = [Symbol]
data Mode = Strict | Weak; data Rule = Mode Word Word; type SRS = [Rule]
```

where lists are built-in (with their usual Haskell definition). A symbol is represented as a binary string. The length of that string is known at run-time (when the implementation sees the signature of the rewrite system) but not at compile-time.

We often access information that belongs to a symbol. We use binary trees where the symbol encodes a path from the root to a leaf

```haskell
data Tree a = Leaf a | Branch (Tree a) (Tree a)
get :: Tree a -> Symbol -> a
get t p = case p of
  [] -> case t of Leaf x -> x
  x:p' -> case t of Branch l r ->
    get (case x of False -> l ; True -> r) p'
```

Symbols will be labelled according to an unknown model, but we know the size of the model. We represent a labelled symbol as the concatenation of a list of known booleans (the original symbol) with a list (of known length) of unknown booleans (the label).

Note that the pattern matches on `p` and on `t` can be resolved at run-time (when generating the SAT formula), but the pattern match on `x` has to be encoded (since `x` depends on the model, which is determined by the solver).

We also represent finite algebras (of size $2^n$) the same way: the domain is the set of bit strings of length `n`, and for each symbol, we have a function from domain to domain:

```haskell
type Model = Tree Func; type Func = Tree Value; type Value = [Bool]
```

We compose functions (and give the most general type):

```haskell
timesF :: Tree a -> Func -> Tree a
timesF f g = case g of
  Leaf w -> Leaf (get f w)
  Branch l r -> Branch (timesF f l) (timesF f r)
```

Given an algebra, we compute the labelled version of rewrite system, as a list of systems (for each original rule, one sub-system containing all its labelled versions). Also, each side of each rule is annotated with its value in the model. These values will be constrained to be equal.

```haskell
labelled :: SRS -> Model -> [ [ ((Value,Value), Rule) ] ]
labelled srs mod =
  let ks = keys ( leftmost mod )
  labelRule u k = case u of
    (l,r) -> case labelledW mod l k of
      ( ltop, l' ) -> case labelledW mod r k of
        ( rtop, r' ) -> ((ltop,rtop),(l',r'))
    in map ( \ u -> map ( \ k -> labelRule u k ) ks ) srs
```
We omit details on interpretations, and come back to the lexicographic comparison mentioned in the introduction:

```
data Comp = Greater | GreaterEquals | None
comp :: Interpretation -> Rule -> Comp
comps :: [ Interpretation ] -> Rule -> Comp
comps ints u = lexi (map ( \ i -> comp i u) ints )
lexi :: [Comp] -> Comp
lexi cs = case cs of
  [] -> GreaterEquals
  c : cs' -> case c of
    Greater -> Greater; GreaterEquals -> lexi cs'; None -> None
```

This shows the expressiveness of the CO4 language. — The main constraint is

```
data Label = Label Model [ Interpretation ] [ Bool ]
constraint :: SRS -> Label -> Bool
where constraint srs (Label mod ints flags)
  is True iff
    mod
  is a model for
    srs
  and
    the lexicographic combination of
      ints
  is compatible with the labelled system, and removes
    the flagged rules (of the original system) completely, and at least one rule is flagged.
```

## Performance

We describe work-in-progress, so we don’t have complete performance data on a larger set of problems. We give an example (that shows the power of the method) and a comparison (that shows the quality of the SAT compilation).

▶ Example 2. Our implementation produces this termination proof for $a^2 b^2 \rightarrow b^3 a^3$:

```
matchbox ~/tpdb/tpdb-4.0/SRS/Zantema/z001.srs -l2,2 --dim 1 --bits 4 --nat
# 2 bits for model values, 2 interpretations for labelled system
CNF finished (#variables: 21483, #clauses: 82956)
Solver finished in 4.152 seconds (result: True)

model: 3b0 2b1 3b2 1b3
       2a0 0a1 0a2 0a3

labelled system: a0 a1 b3 b0 -> b1 b3 b2 a0 a2 a0
                    a0 a1 b3 b2 -> b1 b3 b0 a2 a0 a2
                    a0 a3 b2 b1 -> b1 b3 b0 a2 a0 a1
                    a0 a2 b1 b3 -> b1 b3 b0 a2 a0 a3

natural matrix interpretation 1 natural matrix interpretation 2
entral matrix interpretation 1 [b0,x -> [1] * x + [0]] [b0,x -> [1] * x + [1]]
,[b2,x -> [3] * x + [1]] [b2,x -> [5] * x + [9]]
,[b1,x -> [1] * x + [4]] [b1,x -> [1] * x + [0]]
,[b3,x -> [1] * x + [4]] [b3,x -> [1] * x + [4]]
,[a0,x -> [1] * x + [0]] [a0,x -> [1] * x + [0]]
,[a2,x -> [1] * x + [0]] [a2,x -> [3] * x + [0]]
,[a1,x -> [3] * x + [7]] [a1,x -> [1] * x + [0]]
,[a3,x -> [1] * x + [0]] [a3,x -> [1] * x + [1]]
```
Example 3. We can mimick Jambox’ behaviour by restricting the number of interpretations for the labelled system to 1. With a model of size $2^3$, we get a termination proof (details omitted)

```
matchbox ~/tpdb/tpdb-4.0/SRS/Zantema/z001.srs -l3,1 -d1 -b3 --nat
CNF finished (#variables: 23876, #clauses: 91447)
Solver finished in 2.282 seconds (result: True)
```

Jambox’ formula (for the same parameters) has 33492 variables and 232683 clauses. That means that our CO4 compiler produced a SAT encoding that is comparable to a manual encoding.

6 Extensions

Our implementation additionally allows for each removal step for the labelled system to use arctic or natural matrix interpretation, or lexicographic path order with argument filtering, and also to reverse rules or not—where all these options are encoded in the program, and thus chosen by the solver.

Example 4. This gives one of the shorter termination proofs for Zantema’s system:

```
matchbox ~/tpdb/tpdb-4.0/SRS/Zantema/z001.srs -l2,1 -d1 -b1 --lpo
CNF finished (#variables: 4752, #clauses: 17075)
Solver finished in 1.811 seconds (result: True)
```

Here we compare (with respect to the path order) after applying the morphism (argument filter) that deletes some symbols.

References