Frames in Hilbert C*-modules and stabilization theorems

M. Frank, joint work with David R. Larson (1997-2002)

ESI12 - Modern Methods of Time-Frequency Analysis II

Wien, Nov 19-21, 2012

Motivation

Search for inner structural characterizations of Hilbert C*-modules

Search for common roots of former particular findings of quasi-bases, generating sets, Hilbert bases etc.

Stabilization of Hilbert C*-modules – a particular way to project Hilbert bases into some Hilbert C*-modules

Search for theoretical concepts …

Hilbert C*-modules and frames

A pre-Hilbert C*-module M over a C*-algebra A is a (left) A-module equipped with an A-valued inner product $<\cdot,\cdot>: M\times M \to A$, A-linear in the first argument, with $<x,x>=0$ iff $x=0$, $<x,y>=<y,x>^*$. Norm: $||<\cdot,\cdot>||^{1/2}$.

Def.: (F, L:99-02) A sequence of elements $\{x_i\}_{i \text{ in } I}$ of a Hilbert A-module M over a unital C*-algebra A is a *standard* frame for M if there are two constants $A,B>0$ such that the inequality

$$A\cdot<x,x> \leq \sum_{i \text{ in } I} <x,x_i><x_i,x> \leq B\cdot<x,x>$$

holds for any x *in* M, and the sum converges with respect to the *norm* topology.

(**Bakić**, **Guljaš:02**)

Every Hilbert C*-module over C*-algebras of compact operators contains orthonormal (Parseval) module frames.

For non-standard frames weaker kinds of convergence have to be considered which are derived from the weak*-topology on A^{**} (or from order convergence).

(**Paschke:73**, **Hamana:92** / **F:95**)

Self-dual Hilbert W*-modules and self-dual Hilbert C*-modules over monotone complete C*-algebras always contain orthonormal (Parseval) module frames of that kind.

In case A is unital, M is (algebraically) finitely generated if there exists a finite set $\{ x_i \}_{i \text{ in } \mathbf{N}}$ in M such that $x = \Sigma_i a_i x_i$ for every x in M, some $\{ a_i \}$ in A.

In case A is unital, M is countably generated if there exists a countable set $\{ x_i \}_{i \text{ in } \mathbf{N}}$ in M such that the set of all finite A-linear combinations $\{\Sigma_i a_i x_i : \{ a_i \} \text{ in } A\}$ is norm-dense in M.

Example: $M=C_0((0,1])$ is countably generated over $A=C([0,1])$, $\{ f_1(x)=x \}$. (Weierstraß)

Example: $E: A \to B \subseteq A$ – a conditional expectation,

There are three definitions for a conditional expectation to be of finite index:

1.) There exists $K \geq 1$ such that $K \cdot E - idA \geq 0$.

2.) There exists $L \geq 1$ such that $L \cdot E - idA >> 0$.

3.) There exist elements $\{ a_1, ..., a_n \} \in A$ such that

$$x = \Sigma_{k=1}^{n} a_k \cdot E(a_k^* x) = \Sigma_{k=1}^{n} E(x a_k^*) \cdot a_k$$

for every x *in* A.

There exist examples fulfilling (2), but not (3).

H. Kosaki, 1986 / M. Pimsner , S. Popa, 1986 / M. Baillet, Y. Denizeau, J.-F. Havet, 1988 / Y. Watatani, 1990 / S. Popa, 1995 (1998) / others

Links between the frame theories

Locally trivial Hilbert bundles $\{\xi, X, E, p\}$ over compact Hausdorff spaces X can be identified homeomorphically by their Hilbert $C(X)$-modules of all continuous sections $\Gamma(\xi)$. (↗ duality)

Serre:57, Swan:62, Dixmier-Douady:63, Hofmann-Takahashi-Dupré-Varela:71-74 / Elliott-Kawamura:03+08, others.

Links between (wavelet and) Gabor (Weil-Heisenberg) frame theory and projective Hilbert C*-modules over noncommutative tori.

Rieffel, Packer:97/01-08, Wood:01 / Luef:03-09, others.

Further links into Functional Analysis

Ternary rings of operators (TROs), i.e. linear subspace E of $End_{\mathbb{C}}(H, K)$ satisfying ST^*R *in* E for all S, T, R *in* E.
(↗ duality)

Zettl:83, others.

JB*-triples (with some additional A-module structure) as a slightly generalized version of ternary C*-rings of operators

Neal, Russo:02, others.

A – a unital C*-algebra

$M=A^n$ – all n-tuples with entries of A

$$<a,b> = \Sigma_{i=1}^{n} a_i b_i^*$$

for $a=(a_1, ...,a_n)$, $b=(b_1, ... ,b_n)$ in M.

Theorem: (Dupré, Fillmore:81)

Every algebraically finitely generated Hilbert A-module M is an orthogonal summand of some free Hilbert A-module A^n for $n < \infty$.

H – a Hilbert space, A – a σ-unital C*-algebra

$A \odot H$ – the algebraic tensor product,

$$\langle a \odot h, b \odot g \rangle = a \langle h,g \rangle_H b^*$$

(a,b *in* A, h,g *in* H) becomes a pre-Hilbert A-module, with norm-completion $M = A \otimes H$.

$A^n \simeq A \otimes \mathbf{C}^n$ for any n *in* N.

$l_2(A) \simeq A \otimes l_2$, alternative description:

$$l_2(A) = \{ a=\{a_i\}_{i \text{ in } \mathbf{N}} : \Sigma_{j=1}^{\infty} a_j a_j^* \text{ converges w.r.t. } |.|_A \}$$

with inner product $\langle a,a \rangle = \Sigma_{j=1}^{\infty} a_j a_j^*$.

Theorem: (Kasparov:80) Stabilization Theorem

Every countably generated Hilbert A-module M over a σ-unital C*-algebra A possesses an embedding as an orthogonal summand of $l_2(A)$ in such a way that the orthogonal complement is isometrically isomorphic to $l_2(A)$ again, i.e. $M \oplus l_2(A) = l_2(A)$.

Theorem: (Goswami:09) Stabilization Theorem

An arbitrary (not necessarily countably generated) Hilbert G-A-module M on a (unital) G-C^*-algebra A admits an equivariant embedding into a trivial G-A-module, provided G is a compact Lie group and its strongly continuous action on A is ergodic.

Lemma:

For any A-linear projection P on the Hilbert A-modules A^n or $l_2(A)$ the image of every orthogonal basis of A^n or of $l_2(A)$ is a standard Parseval module frame in $P(A^n)$ or in $P(l_2(A))$, respectively.

Theorem: (Hanfeng Li:08)

For every infinite-dimensional commutative C*-algebra A there exists a Hilbert A-module which does not admit any standard module frame.

Consequently, they do not admit stabilization.

Raeburn,Thompson:02– extension to non-σ-unital case

A – a (non-unital) C*-algebra, H – a Hilbert A-module

$M(A)$ – the multiplier algebra of A

$M(H) := End_A^*(A,H)$ – a Hilbert $M(A)$-module

H is embeddable as a closed, „strictly“ dense $M(A)$-submodule of $M(H)$.

A Hilbert A-module H is countably generated in $M(H)$ if there is a sequence $\{h_i\}_{i \text{ in } \mathbf{N}}$ in $M(H)$ such that the elements $\{ah_i : a \text{ in } A\}$ span a norm-dense A-submodule of H.

→ The frame and stabilization theory works again ...

A – a unital C*-algebra

M – finitely/countably generated Hilbert A-module

$\{ x_i \}_i$ – standard frame in M

The frame transform θ of the frame $\{ x_i \}_i$

$$\theta : M \to l_2(A), \qquad \theta(x) = \{ \langle x, x_i \rangle \}_i$$

where $\{e_i\}_i$ is a fixed orthonormal basis of $l_2(A)$.

– bounded, A-linear, adjointable operator

– $\theta^*(e_i) = x_i$ for all i *in* $\mathbf{N}$

– $\theta(M)$ is an orthogonal summand of $l_2(A)$,
$P: l_2(A) \to \theta(M)$ the respective orthogonal projection.

– for Parseval frames: $P(e_i)=\theta(x_i)$ and θ is an isometry.

The reconstruction formula

$$x= \Sigma_i <x,S(x_i)> x_i$$

holds for every x *in* M in the sense of norm-convergence, where $S=(\theta^*\theta)^{-1}$.

S – the frame operator on M of $\{ x_i \}_i$, $S=(\theta^*\theta)^{-1}$

– bounded, A-linear, positive

$\{ S(x_i) \}_i$ – the canonical dual frame of $\{ x_i \}_i$.

Corollary: Every standard frame of a finitely / countably generated Hilbert *A*-module is a set of generators.

Theorem: (Rieffel:88, F, L:02) Every finite set of algebraic generators of a finitely generated Hilbert *A*-module is a (standard) frame.

Corollary: If $x=\Sigma_i <x,y_i> x_i$ for a different frame $\{ y_i \}_i$ in M , then we have the strong inequality

$$\Sigma_i <x,S(x_i)> <S(x_i),x> \quad < \quad \Sigma_i <x,y_i> <y_i,x> \ .$$

A sequence $\{ x_i \}_i$ *in* M is said to be a standard Riesz basis of M if it is a standard frame and a generating set with the additional property that A-linear combinations $\Sigma_{j \text{ in } S}\, a_j x_j$ with coefficients $\{ a_i \}_i$ *in* A and S *in* $\mathbf{N}$ are equal to zero if and only if in particular every summand $a_j x_j$ equals zero for j *in* S.

Corollary: (F, L :02)

Let A be a unital C*-algebra, M be a finitely or countably generated Hilbert A-module. Suppose that $\{ x_i \}_i$ is a standard Riesz basis for M that is a Parseval frame. Then $\{ x_i \}_i$ is an orthonormal basis with the additional property that $<x_i,x_i> = <x_i,x_i>^2$ any i *in* $\mathbf{N}$. The converse assertion holds too.

Theorem: (F, L :02)

Let $\{ x_i \}_i$ be a standard frame of a finitely or countably generated Hilbert A-module M. Then $\{ x_i \}_i$ is the image of a standard Parseval frame $\{ y_i \}_i$ of M under an invertible adjointable bounded A-linear operator T on M. (E.g., T can be chosen to be positive and equal to the inverted square root of $\theta^*\theta$.)

Conversely, the image of a standard Parseval frame $\{ x_i \}_i$ of M under an invertible adjointable bounded A-linear operator T on M is a standard frame of M.

The frame $\{ x_i \}_i$ is a set of generators of M as an Hilbert A-module.

Corollary: (F, L :02)

Let $\{ x_i \}_i$ be a standard Riesz basis of a finitely or countably generated Hilbert A-module M. Then $\{ x_i \}_i$ is the image of a standard Parseval frame $\{ y_i \}_i$ and orthonormal basis of M under an invertible adjointable bounded A-linear operator T on M, i.e. $<y_j,y_j> = <y_j,y_j>^2$ for any j *in* $\mathbf{N}$. (And conversely.)

If a Hilbert A-module M contains a standard Riesz basis, then M contains an orthonormal basis $\{ x_i \}_i$ with the frame property $x = \Sigma_j <x,x_j>x_j$ for every element x *in* M.

Definition:

Two frames $\{ x_i \}_i$, $\{ y_i \}_i$ are unitarily equivalent / similar if there exists a unitary / invertible adjointable A-linear operator T such that $T(x_i)=y_i$ for every i *in* $\mathbf{N}$.

Theorem: (F, L :02)

Let A be a unital C*-algebra and $\{ x_j \}_j$ and $\{ y_j \}_j$ be standard Parseval frames of Hilbert A-modules M_1 and M_2, respectively.

Then the frames $\{ x_j \}_j$ and $\{ y_j \}_j$ are unitarily equivalent if and only if their frame transforms θ_1 *and* θ_2 have the same range in $l_2(A)$, if and only if the sums $\Sigma_j a_j x_j$ and $\Sigma_j a_j y_j$ equal zero for exactly the same Banach A-submodule of sequences $\{ a_j \}_j$ of $l_2(A)'$.

Theorem: (F, L :02)

Let A be a unital C*-algebra and $\{ x_j \}_j$ and $\{ y_j \}_j$ be standard frames of Hilbert A-modules M_1 and M_2, respectively.

Then the frames $\{ x_j \}_j$ and $\{ y_j \}_j$ are similar if and only if their frame transforms θ_1 *and* θ_2 have the same range in $l_2(A)$, if and only if the sums $\Sigma_j a_j x_j$ and $\Sigma_j a_j y_j$ equal zero for exactly the same Banach A-submodule of sequences $\{ a_j \}_j$ of $l_2(A)'$.

Proposition: (F, L :02)

Two different standard alternate dual frames of a given frame are not similar or unitarily equivalent.

Alternative concepts of frames

Operator-valued frames:

Let H and H_0 be Hilbert spaces. A collection $\{ A_j \}_{j \in J}$ of operators $A_j \in B(H,H_0)$ indexed by J is called an operator-valued frame on H with range in H_0 if the series $S_A := \Sigma_{j \in J} A^*_j A_j$ converges in the strong operator topology to a positive bounded invertible operator S_A.

Rank-one operators give the classical frames.

V. Kaftal, D. R. Larson, Shuang Zhang:09, others

Alternative concepts of frames

g-frames:

A generalization of operator-valued frames, for which every frame element A_j has its own target Hilbert space H_j.

W. Sun:06-08, Xiang-Chun Xiao, Xiao-Ming Zeng:10, A. Khosravi et al. (Iranian School):08-12, and others.

Thank you for your attention.

Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig

Michael Frank
Fakultät Informatik, Mathematik und Naturwissenschaften

Wien, 21.11.2012

ANHANG

A **JB*-triple** is a Banach space X with a product

$$D(x, y)z = \{x\ y\ z\}$$

which is linear in the outer variables, conjugate linear in the middle variable, is commutative i.e. $\{x\ y\ z\} = \{z\ y\ x\}$, satisfies an associativity condition

$$[D(x, y), D(a, b)] = D(\{x\ y\ a\}, b) - D(a, \{b\ x\ y\})$$

and has the topological properties that

(i) $||D(x, x)|| = ||x||^2$ for any x *in* X

(ii) $D(x, x)$ is hermitian (in the sense that $||e^{\mathbf{i}tD(x,x)}|| = 1$), has positive spectrum in the Banach algebra $B(X)$.

Definition: (Zettl,1983)

A ternary C*-ring $\{E, (.,.,.), ||.||\}$ consists of a complex Banach space $\{E, ||.||\}$ and a ternary operation $(.,.,.): E \times E \times E \to E$ such that for any v,w,x,y,z in E and λ *in* $\mathbf{C}$ one has

(i) $(.,.,.)$ is linear in the first and third variable, and anti-linear in the second variable.

(ii) $((v,w,x),y,z) = (v,(y,x,w),z) = (v,w,(x,y,z))$

(iii) $||(x,y,z)|| \leq ||x||\ ||y||\ ||z||$

(iv) $||(x,x,x)|| = ||x||^3$

→ Gives a link between Hilbert C*-modules and operator modules, operator spaces.

A ternary ring of operators (TRO) between two complex Hilbert spaces H and K is a linear subspace E of $End_{\mathbb{C}}(H, K)$ satisfying ST^*R in E for all S,T,R in E.

Theorem: (Zettl,1983)

For each ternary C*-ring $\{E, (.,.,.), \|.\|\}$ there exists exactly one operator $T: E \to E$ such that (i) $T^2=1$; (ii) $T((x,y,z))=(T(x),y,z)=(x,T(y),z)=(x,y,T(z))$ for any x,y,z in E; (iii) $\{E, T \circ (.,.,.), \|.\|\}$ is a ternary C*-ring isomorphic to a norm-closed TRO.

Theorem: (Zettl,1983)

Norm-closed TRO‘s are exactly Hilbert C*-modules, and vice versa.

Categorical equivalence between Hilbertian A-modules over finite von Neumann algebras A with a faithful normal trace state, and self-dual Hilbert A-modules.

Applications to L^2-invariants possible.

V. Mathai, A. Carey 1990-2000, W. Lück, M. Rothenberg, 1991-2000, M. Farber, 1995-2001, J. Lott, 1996-2000, D. Burghelea, L. Friedlander, T. Kappeler, P. McDonald, 1996-2000, Th. Schick, 1996-2000, A. Thom, 2008-2009, Ch. Wahl, 2008, others.