Frames in Hilbert C*-modules and stabilization theorems

M. Frank, joint work with David R. Larson (1997-2002)
Motivation

Search for inner structural characterizations of Hilbert C*-modules

Search for common roots of former particular findings of quasi-bases, generating sets, Hilbert bases etc.

Stabilization of Hilbert C*-modules – a particular way to project Hilbert bases into some Hilbert C*-modules

Search for theoretical concepts …
Hilbert C*-modules and frames

A pre-Hilbert C*-module $M$ over a C*-algebra $A$ is a (left) $A$-module equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : M \times M \to A$, $A$-linear in the first argument, with $\langle x, x \rangle = 0$ iff $x = 0$, $\langle x, y \rangle = \langle y, x \rangle^*$. Norm: $\| \langle \cdot, \cdot \rangle \|^{1/2}$.

Def.: (F, L:99-02) A sequence of elements $\{x_i\}_{i \in I}$ of a Hilbert $A$-module $M$ over a unital C*-algebra $A$ is a standard frame for $M$ if there are two constants $A, B > 0$ such that the inequality

$$A \cdot \langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq B \cdot \langle x, x \rangle$$

holds for any $x$ in $M$, and the sum converges with respect to the norm topology.
(Bakić, Guljaš:02)

Every Hilbert C*-module over C*-algebras of compact operators contains orthonormal (Parseval) module frames.

For non-standard frames weaker kinds of convergence have to be considered which are derived from the weak*-topology on $A^{**}$ (or from order convergence).

(Paschke:73, Hamana:92 / F:95)

Self-dual Hilbert W*-modules and self-dual Hilbert C*-modules over monotone complete C*-algebras always contain orthonormal (Parseval) module frames of that kind.
In case $A$ is unital, $M$ is (algebraically) finitely generated if there exists a finite set $\{ x_i \}_{i \in \mathbb{N}}$ in $M$ such that $x = \sum_i a_i x_i$ for every $x \in M$, some $\{ a_i \}$ in $A$.

In case $A$ is unital, $M$ is countably generated if there exists a countable set $\{ x_i \}_{i \in \mathbb{N}}$ in $M$ such that the set of all finite $A$-linear combinations $\{ \sum_i a_i x_i : \{ a_i \} \in A \}$ is norm-dense in $M$.

**Example:** $M = C_0((0,1])$ is countably generated over $A = C([0,1])$, $\{ f_1(x) = x \}$. (Weierstraß)
Example: \(E: A \rightarrow B \subseteq A\) – a conditional expectation,

There are three definitions for a conditional expectation to be of finite index:

1.) There exists \(K \geq 1\) such that \(K \cdot E - idA \geq 0\).
2.) There exists \(L \geq 1\) such that \(L \cdot E - idA > > 0\).
3.) There exist elements \(\{a_1, ..., a_n\} \in A\) such that

\[x = \sum_{k=1}^{n} a_k \cdot E(a_k^*x) = \sum_{k=1}^{n} E(xa_k^*) \cdot a_k\]

for every \(x \in A\).

There exist examples fulfilling (2), but not (3).

Links between the frame theories

Locally trivial Hilbert bundles $\{\xi, X, E, p\}$ over compact Hausdorff spaces $X$ can be identified homeomorphically by their Hilbert $C(X)$-modules of all continuous sections $\Gamma(\xi)$. (↑ duality)


Links between (wavelet and) Gabor (Weil-Heisenberg) frame theory and projective Hilbert $C^*$-modules over noncommutative tori.

Rieffel, Packer:97/01-08, Wood:01 / Luef:03-09, others.
Further links into Functional Analysis

Ternary rings of operators (TROs), i.e. linear subspace $E$ of $\text{End}_{\mathbb{C}}(H, K)$ satisfying $ST^*R \in E$ for all $S, T, R \in E$. (↑ duality)

Zettl:83, others.

JB*-triples (with some additional $A$-module structure) as a slightly generalized version of ternary $C^*$-rings of operators

Neal, Russo:02, others.
$A$ – a unital C*-algebra

$M = A^n$ – all $n$-tuples with entries of $A$

\[ <a, b> = \sum_{i=1}^{n} a_i b_i^* \]

for $a = (a_1, \ldots, a_n), \ b = (b_1, \ldots, b_n)$ in $M$.

**Theorem: (Dupré, Fillmore:81)**

Every algebraically finitely generated Hilbert $A$-module $M$ is an orthogonal summand of some free Hilbert $A$-module $A^n$ for $n < \infty$. 
Frames in Hilbert $\mathrm{C}^*$-modules and stabilization theorems

$H$ – a Hilbert space, $A$ – a $\sigma$-unital $\mathrm{C}^*$-algebra

$A \odot H$ – the algebraic tensor product,

$$<a \odot h, b \odot g> = a <h, g>_H b^*$$

$(a,b \in A, h,g \in H)$ becomes a pre-Hilbert $A$-module, with norm-completion $M = A \otimes H$.

$A^n = A \otimes C^n$ for any $n \in \mathbb{N}$.

$l_2(A) = A \otimes l_2$, alternative description:

$$l_2(A) = \{ a = \{a_i\}_{i \in \mathbb{N}} : \sum_{j=1}^{\infty} a_j a_j^* \text{ converges w.r.t. } |.|_A \}$$

with inner product $<a,a> = \sum_{j=1}^{\infty} a_j a_j^*$. 

Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig

Michael Frank
Fakultät Informatik, Mathematik und Naturwissenschaften

21.11.2012/10
Theorem: (Kasparov:80) Stabilization Theorem

Every countably generated Hilbert $A$-module $M$ over a $\sigma$-unital C*-algebra $A$ possesses an embedding as an orthogonal summand of $l_2(A)$ in such a way that the orthogonal complement is isometrically isomorphic to $l_2(A)$ again, i.e. $M \oplus l_2(A) = l_2(A)$.

Theorem: (Goswami:09) Stabilization Theorem

An arbitrary (not necessarily countably generated) Hilbert $G$-$A$-module $M$ on a (unital) $G$-$C^*$-algebra $A$ admits an equivariant embedding into a trivial $G$-$A$-module, provided $G$ is a compact Lie group and its strongly continuous action on $A$ is ergodic.
Lemma:
For any $A$-linear projection $P$ on the Hilbert $A$-modules $A^n$ or $l_2(A)$ the image of every orthogonal basis of $A^n$ or of $l_2(A)$ is a standard Parseval module frame in $P(A^n)$ or in $P(l_2(A))$, respectively.

Theorem: (Hanfeng Li:08)
For every infinite-dimensional commutative C*-algebra $A$ there exists a Hilbert $A$-module which does not admit any standard module frame. Consequently, they do not admit stabilization.
Raeburn, Thompson:02 – extension to non-σ-unital case

\( A \) – a (non-unital) \( C^* \)-algebra, \( H \) – a Hilbert \( A \)-module

\( M(A) \) – the multiplier algebra of \( A \)

\( M(H) := \text{End}_{A^*}(A,H) \) – a Hilbert \( M(A) \)-module

\( H \) is embeddable as a closed, „strictly“ dense \( M(A) \)-submodule of \( M(H) \).

A Hilbert \( A \)-module \( H \) is countably generated in \( M(H) \) if there is a sequence \( \{h_i\}_{i \in \mathbb{N}} \) in \( M(H) \) such that the elements \( \{ah_i : a \in A\} \) span a norm-dense \( A \)-submodule of \( H \).

\( \rightarrow \) The frame and stabilization theory works again ...
Frames in Hilbert C*-modules and stabilization theorems

\[ A \] – a unital C*-algebra  
\[ M \] – finitely/countably generated Hilbert A-module  
\[ \{ x_i \} \] – standard frame in \( M \)

The frame transform \( \theta \) of the frame \( \{ x_i \} \)

\[ \theta : M \to l_2(A), \quad \theta(x) = \{ \langle x, x_i \rangle \} \]

where \( \{ e_i \} \) is a fixed orthonormal basis of \( l_2(A) \).

\( \theta \) is bounded, \( A \)-linear, adjointable operator  
\( \theta^*(e_i) = x_i \) for all \( i \) in \( \mathbb{N} \)  
\( \theta(M) \) is an orthogonal summand of \( l_2(A) \), 
\[ P : l_2(A) \to \theta(M) \] the respective orthogonal projection.
– for Parseval frames: \( P(e_i) = \theta(x_i) \) and \( \theta \) is an isometry.

The reconstruction formula

\[
x = \sum_i \langle x, S(x_i) \rangle x_i
\]

holds for every \( x \) in \( M \) in the sense of norm-convergence, where \( S = (\theta^* \theta)^{-1} \).

\( S \) – the frame operator on \( M \) of \( \{ x_i \}_i \), \( S = (\theta^* \theta)^{-1} \)

– bounded, A-linear, positive

\( \{ S(x_i) \}_i \) – the canonical dual frame of \( \{ x_i \}_i \).
Corollary: Every standard frame of a finitely / countably generated Hilbert $A$-module is a set of generators.

Theorem: (Rieffel:88, F, L:02) Every finite set of algebraic generators of a finitely generated Hilbert $A$-module is a (standard) frame.

Corollary: If $x=\sum_i <x,y_i> x_i$ for a different frame $\{ y_i \}_i$ in $M$, then we have the strong inequality

$$\sum_i <x,S(x_i)> <S(x_i),x> < \sum_i <x,y_i> <y_i,x> .$$
A sequence \( \{ x_i \}_i \) in \( M \) is said to be a standard Riesz basis of \( M \) if it is a standard frame and a generating set with the additional property that \( A \)-linear combinations \( \sum_{j \in S} a_j x_j \) with coefficients \( \{ a_i \}_i \) in \( A \) and \( S \) in \( N \) are equal to zero if and only if in particular every summand \( a_j x_j \) equals zero for \( j \) in \( S \).

**Corollary: (F, L :02)**

Let \( A \) be a unital C*-algebra, \( M \) be a finitely or countably generated Hilbert \( A \)-module. Suppose that \( \{ x_i \}_i \) is a standard Riesz basis for \( M \) that is a Parseval frame. Then \( \{ x_i \}_i \) is an orthonormal basis with the additional property that \( \langle x_i, x_i \rangle = \langle x_i, x_i \rangle^2 \) any \( i \) in \( N \). The converse assertion holds too.
Theorem: (F, L :02)

Let \( \{ x_i \}_i \) be a standard frame of a finitely or countably generated Hilbert \( A \)-module \( M \). Then \( \{ x_i \}_i \) is the image of a standard Parseval frame \( \{ y_i \}_i \) of \( M \) under an invertible adjointable bounded \( A \)-linear operator \( T \) on \( M \). (E.g., \( T \) can be chosen to be positive and equal to the inverted square root of \( \theta^*\theta \).)

Conversely, the image of a standard Parseval frame \( \{ x_i \}_i \) of \( M \) under an invertible adjointable bounded \( A \)-linear operator \( T \) on \( M \) is a standard frame of \( M \). The frame \( \{ x_i \}_i \) is a set of generators of \( M \) as an Hilbert \( A \)-module.
Corollary: (F, L :02)

Let \( \{ x_i \}_i \) be a standard Riesz basis of a finitely or countably generated Hilbert \( A \)-module \( M \). Then \( \{ x_i \}_i \) is the image of a standard Parseval frame \( \{ y_i \}_i \) and orthonormal basis of \( M \) under an invertible adjointable bounded \( A \)-linear operator \( T \) on \( M \), i.e. \( <y_j, y_j> = <y_j, y_j>^2 \) for any \( j \) in \( \mathbb{N} \). (And conversely.)

If a Hilbert \( A \)-module \( M \) contains a standard Riesz basis, then \( M \) contains an orthonormal basis \( \{ x_i \}_i \) with the frame property \( x = \Sigma_j <x, x_j>x_j \) for every element \( x \) in \( M \).
Definition:

Two frames \( \{ x_i \}_i \), \( \{ y_i \}_i \) are **unitarily equivalent / similar** if there exists a unitary / invertible adjointable \( A \)-linear operator \( T \) such that \( T(x_i) = y_i \) for every \( i \ in \mathbb{N} \).

Theorem: (F, L :02)

Let \( A \) be a unital C*-algebra and \( \{ x_j \}_j \) and \( \{ y_j \}_j \) be standard Parseval frames of Hilbert \( A \)-modules \( M_1 \) and \( M_2 \), respectively.

Then the frames \( \{ x_j \}_j \) and \( \{ y_j \}_j \) are unitarily equivalent if and only if their frame transforms \( \theta_1 \) and \( \theta_2 \) have the same range in \( l_2(A) \), if and only if the sums \( \Sigma_j a_j x_j \) and \( \Sigma_j a_j y_j \) equal zero for exactly the same Banach \( A \)-submodule of sequences \( \{ a_j \}_j \) of \( l_2(A)' \).
Theorem: (F, L :02)

Let \( A \) be a unital \( C^* \)-algebra and \( \{ x_j \}_j \) and \( \{ y_j \}_j \) be standard frames of Hilbert \( A \)-modules \( M_1 \) and \( M_2 \), respectively.

Then the frames \( \{ x_j \}_j \) and \( \{ y_j \}_j \) are similar if and only if their frame transforms \( \theta_1 \) and \( \theta_2 \) have the same range in \( l_2(A) \), if and only if the sums \( \sum_j a_j x_j \) and \( \sum_j a_j y_j \) equal zero for exactly the same Banach \( A \)-submodule of sequences \( \{ a_j \}_j \) of \( l_2(A)' \).

Proposition: (F, L :02)

Two different standard alternate dual frames of a given frame are not similar or unitarily equivalent.
Alternative concepts of frames

Operator-valued frames:
Let $H$ and $H_0$ be Hilbert spaces. A collection $\{A_j\}_{j \in J}$ of operators $A_j \in B(H,H_0)$ indexed by $J$ is called an operator-valued frame on $H$ with range in $H_0$ if the series $S_A := \sum_{j \in J} A_j^* A_j$ converges in the strong operator topology to a positive bounded invertible operator $S_A$.

Rank-one operators give the classical frames.

V. Kaftal, D. R. Larson, Shuang Zhang:09, others
Alternative concepts of frames

\textbf{\textit{g}-frames:}}

A generalization of operator-valued frames, for which every frame element $A_j$ has its own target Hilbert space $H_j$.

W. Sun:06-08, Xiang-Chun Xiao, Xiao-Ming Zeng:10, A. Khosravi et al. (Iranian School):08-12, and others.
Thank you for your attention.
ANHANG
A JB*-triple is a Banach space $X$ with a product

$$D(x, y)z = \{x \ y \ z\}$$

which is linear in the outer variables, conjugate linear in the middle variable, is commutative i.e. $\{x \ y \ z\} = \{z \ y \ x\}$, satisfies an associativity condition

$$[D(x, y), D(a, b)] = D(\{x \ y \ a\}, b) - D(a, \{b \ x \ y\})$$

and has the topological properties that

(i) $||D(x, x)|| = ||x||^2$ for any $x$ in $X$

(ii) $D(x, x)$ is hermitian (in the sense that $||e^{itD(x,x)}|| = 1$), has positive spectrum in the Banach algebra $B(X)$. 
Definition: (Zettl, 1983)

A ternary C*-ring \( \{E, (.,.,.), ||.||\} \) consists of a complex Banach space \( \{E, ||.||\} \) and a ternary operation \( (.,.,.): E \times E \times E \to E \) such that for any \( v, w, x, y, z \) in \( E \) and \( \lambda \) in \( \mathbb{C} \) one has

(i) \( (.,.,.) \) is linear in the first and third variable, and anti-linear in the second variable.
(ii) \( ((v, w, x), y, z) = (v, (y, x, w), z) = (v, w, (x, y, z)) \)
(iii) \( ||(x, y, z)|| \leq ||x|| \, ||y|| \, ||z|| \)
(iv) \( ||(x, x, x)|| = ||x||^3 \)

→ Gives a link between Hilbert C*-modules and operator modules, operator spaces.
A ternary ring of operators (TRO) between two complex Hilbert spaces $H$ and $K$ is a linear subspace $E$ of $\text{End}_C(H, K)$ satisfying $ST^*R$ in $E$ for all $S, T, R$ in $E$.

**Theorem: (Zettl, 1983)**
For each ternary C*-ring $\{E, (.,.,.), ||.||\}$ there exists exactly one operator $T: E \to E$ such that (i) $T^2=1$; (ii) $T((x,y,z))=(T(x),y,z)=(x,T(y),z)=(x,y,T(z))$ for any $x, y, z$ in $E$; (iii) $\{E, T \circ (.,.,.), ||.||\}$ is a ternary C*-ring isomorphic to a norm-closed TRO.

**Theorem: (Zettl, 1983)**
Norm-closed TRO’s are exactly Hilbert C*-modules, and vice versa.
Categorical equivalence between Hilbertian $A$-modules over finite von Neumann algebras $A$ with a faithful normal trace state, and self-dual Hilbert $A$-modules.

Applications to $L^2$-invariants possible.