# PROBLEM FIELD: APPROXIMATION OF FRAMES BY NORMALIZED TIGHT ONES

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Let $\{x_i\}_i$ be a frame of a Hilbert subspace $K \subseteq H$ of a given (separable) Hilbert space $H$ with upper and lower frame bounds $B$ and $A$. The resulting frame transform is the map $\theta : K \to l_2$ , $\theta(x) = \{\langle x, g_i\rangle\}_i$, and its adjoint operator is $\theta^* : l_2 \to K$ , $\theta^*(e_i) = g_i$, $i \in \mathbb{N}$, for the standard orthonormal basis $\{e_i\}_i$ of $l_2$. Let $S = (\theta^*\theta)^{-1}$ be the frame operator defined on $K$. It is positive and invertible. There exists an orthogonal projection $P : l_2 \to \theta\theta^*(l_2) \subseteq l_2$ onto the range of the frame transform.

**Problem:**

Are there distance measures on the set of frames of all Hilbert subspaces $L$ of $H$ with respect to which a multiple of the normalized tight frame $\{S^{1/2}(x_i)\}_i$ is the closest normalized tight frame to the given frame $\{x_i\}_i$ of the Hilbert subspace $K \subseteq H$, or at least one of the closest normalized tight frames?

If there are other closest normalized tight frames with respect to the selected distance measures, do they span the same Hilbert subspaces of $H$? If not, how are the positions of the subspaces with respect to $K \subseteq H$?

To obtain at least partial results authors usually have applied some additional restrictions to the set of frames to be considered: (i) resort to similar frames, (ii) resort to the case $K = L = H$, (iii) resort to special classes of frames like Gabor (Weyl-Heisenberg) or wavelet frames, and others. So one goal might be to lessen the restrictions in the suppositions.

We would like to list some existing results from [1], [3] and [4] to give a flavor of the existing successful approaches and to outline the wide open field of research to be filled. From recent correspondences with R. Balan we know about new findings of him and Z. Landau to be published in the near future ([2]).

First recall the major results by R. Balan ([1]): The frame $\{x_i\}_i$ of the Hilbert space $H$ is said to be *quadratically close* to the frame $\{y_i\}_i$ of $H$ if there exists a non-negative number $C$ such that the inequality

$$\left\|\sum\nolimits_i c_i(x_i - y_i)\right\| \leq C \cdot \left\|\sum\nolimits_i c_i y_i\right\|$$

is satisfied. The infimum of all such constants $C$ is denoted by $c(y, x)$. In general, if $C \geq c(y, x)$ then $C(1 - C)^{-1} \geq c(x, y)$, however this distance measure is not reflexive. Two frames $\{x_i\}_i$ and $\{y_i\}_i$ of a Hilbert space $H$ are said to be *near* if $d(x, y) = \log(\max(c(x, y), c(y, x)) + 1) < \infty$. They are near if and only if they are similar, [1, Th. 2.4]. The distance measure $d(x, y)$ is an equivalence relation and fulfills the triangle inequality.

**Theorem:** (R. Balan, 1997)
For a given frame $\{x_i\}_i$ of $H$ the distance measures admit their infima at

$$\min c(y, x) = \min c(x, y) = \frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}\,,\ \min d(x, y) = \frac{1}{4}(\log(B) - \log(A))\,.$$

These values are achieved by the tight frames

$$\left\{ \frac{\sqrt{A}+\sqrt{B}}{2} S^{1/2}(x_i) \right\}_i \,, \left\{ \frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}} S^{1/2}(x_i) \right\}_i \,, \{ \sqrt[4]{AB} S^{1/2}(x_i) \}_i \,,$$

in the same order as the three measures are listed above. The solution may not be unique, in general, however any tight frame $\{y_i\}_i$ of $H$ that achieves the minimum of one of the three distance measures $c(y,x)$, $c(x,y)$ and $d(x,y)$ is unitarily equivalent to the corresponding solutions listed above. The difference of the connecting unitary operator and the product of minimal distance times either $S^{1/2}$ or $S^{-1/2}$ fulfills a measure-specific operator norm equality.

A second class of examples has been treated by T. R. Tiballi, V. I. Paulsen and the author in 1998 ([3]). The foundations were laid by T. R. Tiballi in his Master Thesis in 1991 ([6]). Therein he was dealing with the symmetric orthogonalization of orthonormal bases of Hilbert spaces in a way that did not make use of the linear independence of the elements. So his techniques have been extendable to the situation of frames giving rise to the symmetric approximation of frames by normalized tight ones.

**Theorem:** (M. Frank, V. I. Paulsen, T. R. Tiballi, 1998)
The operator $(P - |\theta^*|)$ is Hilbert-Schmidt if and only if the sum $\sum_{j=1}^\infty \|\mu_j - x_j\|^2$ is finite for at least one normalized tight frame $\{\mu_i\}_i$ of a Hilbert subspace $L$ of $H$ that is similar to $\{x_i\}_i$. In this situation the estimate

$$\sum_{j=1}^\infty \|\mu_j - x_j\|^2 \geq \sum_{j=1}^\infty \|S^{1/2}(x_j) - x_j\|^2 = \|(P - |\theta^*|)\|_{c_2}^2$$

is valid for every normalized tight frame $\{\mu_i\}_i$ of any Hilbert subspace $L$ of $H$ that is similar to $\{x_i\}_i$. (The left sum can be infinite for some choices of subspaces $L$ and normalized tight frames $\{\mu_i\}_i$ for them.)
Equality appears if and only if $\mu_i = S^{1/2}(x_i)$ for any $i \in \mathbb{N}$. Consequently, *the symmetric approximation of a frame $\{x_i\}_i$ in a Hilbert space $K \subseteq H$* is the normalized tight frame $\{S^{1/2}(x_i)\}_i$ spanning the same Hilbert subspace $L \equiv K$ of $H$ and being similar to $\{x_i\}_i$ via the invertible operator $S^{-1/2}$.

**Remark:** (see Per-Olov Löwdin, 1948/1970)
If $\{x_i\}_i$ is a Riesz basis, then $\{S^{1/2}(x_i)\}_i$ is *the symmetric orthogonalization* of this basis. This is why the denotation 'symmetric approximation' has been selected.

A third approach has been developed by Deguang Han investigating approximation of Gabor (Weyl-Heisenberg) and wavelet frames. His starting point are countable *unitary systems* $\mathcal{U}$ on separable Hilbert spaces that contain the identity operator. In particular, $\mathcal{U}$ is supposed to be *group-like*, i.e. *group*$(\mathcal{U}) \subseteq \mathbb{T}\mathcal{U} = \{\lambda U : \lambda \in \mathbb{T}, U \in \mathcal{U}\}$. A vector $\phi \in H$ is a *complete frame vector* (resp., a *normalized tight frame vector*) for $\mathcal{U}$ if the set $\mathcal{U}\phi := \{U(\phi) : U \in \mathcal{U}\}$ is a frame (resp., a normalized tight frame) of $H$. Two *frame vectors* $\phi, \psi \in H$ are said to be *similar* if the two frames $\mathcal{U}\phi$ and $\mathcal{U}\psi$ are similar frames in $H$. Let $\mathcal{T}(\mathcal{U})$ denote the set of all normalized tight frame vectors of $H$ with respect to the action of $\mathcal{U}$.
As a matter of fact the distance measure used in [3] gives $\sum_{U \in \mathcal{U}} \|U(\xi) - U(\eta)\|^2 = \infty$ if $\mathcal{U}$ is an infinite set and $\xi \neq \eta$. Also, $\mathcal{U}\xi$ and $\mathcal{U}\eta$ are not similar, in general, cf. [1]. So define a vector $\psi \in \mathcal{T}(\mathcal{U})$ to be a *best normalized tight frame (NTF) approximation* for a given complete frame vector $\phi \in H$ of $\mathcal{U}$ if

$$\|\psi - \phi\| := \mathrm{dist}(\phi, \mathcal{T}(\mathcal{U})) := \inf\{\|\eta - \phi\| : \eta \in \mathcal{T}(\mathcal{U})\} \,.$$

**Theorem:** (Deguang Han, 2001)
Let $\mathcal{U}$ be a group-like unitary system acting on a Hilbert space $H$. Let $\phi \in H$ be a complete frame vector for $\mathcal{U}$. Then the vector $S^{1/2}(\phi)$ is the unique best NTF approximation for $\phi$, where $S = (\theta^*\theta)^{-1}$ is the frame operator for $\phi$.

**Theorem:** (Deguang Han, 2001)
Let $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$ be a full-rank lattice and $g$ be a Gabor frame generator associated with $\Lambda$. Then the vector $S^{1/2}(g)$ is the unique best NTF approximation for $g$, where $S$ is the frame operator for $g$. ($S^{1/2}(g)$ is a Gabor frame generator, again.)

Considering the wavelet situation where the generating unitary systems sometimes are not group-like some obstacles are encountered. For example, D. Han found that for an orthonormal wavelet $\mathcal{U}_{D,T}(g)$ the vector $\phi = 1/4 \cdot g$ possesses better NTF approximations than $S^{1/2}(\phi)$. In ongoing discussions of D. Han with I. Daubechies, J. Wexler and M. Bownik examples of wavelet frames have been found for which there does not exist any wavelet-type dual frame. It is unknown whether $\{S^{1/2}(x_i)\}_i$ has always wavelet structure for wavelet frames $\{x_i\}_i$, or not.

**Theorem :** (Deguang Han, 2001)
Suppose, $\phi$ is the generator of a semi-orthogonal wavelet frame, i.e. $\phi_{m,k} \perp \phi_{n,l}$ for $\phi_{m,k} := |\det(M)|^{m/2}\phi(M^m x - k)$ and for any $k, l \in \mathbb{Z}^d$, all $m, n \in \mathbb{Z}$ with $m \neq n$. Denote by $\mathcal{U}_{D,T}$ the unitary system generating the initial wavelet frame. Then there exists a unique normalized tight wavelet frame generated by $\psi$ such that the equality

$$\|\phi - \psi\| = \min\{\|h - \phi\| \,:\, h \in \mathcal{T}(\mathcal{U}_{D.T}),\, h \sim \phi\}$$

holds. Moreover, $\psi = S^{1/2}(\phi)$.

## References


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