# Termination of $\{aa \to cb, bb \to ca, cc \to ba\}$

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The general setting: Let $\succ$ be a well-founded order on a non-empty set $D$, and let $\succeq$ denote its reflexive closure. Let $i : \Sigma \to (D \to D)$ be a mapping which is extended to $\Sigma^*$ by $i(\epsilon)(d) = d$ and $i(xs)(d) = i(x)(i(s)(d))$ for $x \in \Sigma$, $s \in \Sigma^*$, $d \in D$. Further assume strict monotonicity, i.e., for $x \in \Sigma$ and $d, d' \in D$, $d \succ d'$ implies $i(x)(d) \succ i(x)(d')$. For a proof of the following standard result see e.g. Theorem 4 in [Zantema, Termination of string rewriting proved automatically].

**Lemma 1.** *Let $R$ and $S$ be string rewriting systems over alphabet $\Sigma$ such that*

- $i(\ell)(d) \succ i(r)(d)$ *for all* $\ell \to r$ *in* $R$ *and* $d \in D$,
- $i(\ell)(d) \succeq i(r)(d)$ *for all* $\ell \to r$ *in* $S$ *and* $d \in D$.

*Then $R$ is terminating relative to $S$.*

**Proposition 1.** *$Z_{086} = \{aa \to cb, bb \to ca, cc \to ba\}$ is terminating.*

*Proof.* Let $\Sigma = \{a, b, c\}$. Consider the interpretation $i : \Sigma \to (\mathbb{N}^4 \to \mathbb{N}^4)$ with

$$\begin{aligned} i(a)(u,v,x,y) &= (u+y, y, v+2y+2, v),\\ i(b)(u,v,x,y) &= (u+v, 2v+y+1, v, 0),\\ i(c)(u,v,x,y) &= (u+2y, x+y, v+2y+1, 0), \end{aligned}$$

Let $(u,v,x,y) \succ (u',v',x',y')$ if $u > u'$ and $v \geq v'$, $x \geq x'$, $y \geq y'$, and let $\succeq$ be the reflexive closure of $\succ$. Note that $\succ$ is well-founded on $\mathbb{N}^4$, and that $i(x)$ is strictly monotone for each $x \in \Sigma$. As is easily verified,

$$\begin{aligned} i(aa)(u,v,x,y) &= (u+v+y, v, 2v+y+2, y)\\ &\succeq (u+v, v, 2v+y+2, 0) = i(cb)(u,v,x,y),\\ i(bb)(u,v,x,y) &= (u+3v+y+1, 4v+2y+3, 2v+y+1, 0)\\ &\succ (u+2v+y, 2v+2y+2, 2v+y+1, 0) = i(ca)(u,v,x,y),\\ i(cc)(u,v,x,y) &= (u+2y, v+2y+1, x+y+1, 0)\\ &\succeq (u+2y, v+2y+1, y, 0) = i(ba)(u,v,x,y). \end{aligned}$$

By the lemma above, termination of $S = \{aa \to cb, cc \to ba\}$ implies termination of $Z_{086}$. Termination of $S$ is trivial: choose weights $a, c \mapsto 1$ and $b \mapsto 0$. $\square$

To illustrate, here ist the sequence of interpretations for a rewrite sequence starting from $aabb$:

$$\begin{array}{rcrcrcr} a\,(4,3,8,0) & & c\,(4,3,8,0) & & c\,(2,3,5,0) & & b\,(2,3,1,0)\\ a\,(1,0,5,3) & & b\,(4,7,3,0) & & c\,(2,4,3,0) & & a\,(1,1,4,0)\\ b\,(1,3,0,0) & \to_1 & b\,(1,3,1,0) & \to_2 & a\,(0,0,3,1) & \to_3 & a\,(0,0,3,1)\\ b\,(0,1,0,0) & & b\,(0,1,0,0) & & b\,(0,1,0,0) & & b\,(0,1,0,0)\\ (0,0,0,0) & & (0,0,0,0) & & (0,0,0,0) & & (0,0,0,0) \end{array}$$

$$\begin{array}{crcrcr} & b\,(2,3,1,0) & & b\,(1,3,1,0) & & b\,(1,3,1,0)\\ & c\,(1,1,4,0) & & c\,(0,1,3,0) & & b\,(0,1,0,0)\\ \to_4 & b\,(1,3,1,0) & \to_5 & c\,(0,2,1,0) & \to_6 & a\,(0,0,2,0)\\ & b\,(0,1,0,0) & & a\,(0,0,2,0) & & a\,(0,0,2,0)\\ & (0,0,0,0) & & (0,0,0,0) & & (0,0,0,0) \end{array}$$

Note the strict decreases in steps 2 and 5.