# Equivalence of Match-Boundedness, Change-Boundedness and inverse Match-Boundedness for Length-Preserving String Rewriting

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# String Rewriting

*string rewriting system* $R$ over alphabet $\Sigma$ is set of rules, *rule* is pair of words. $R \subseteq \Sigma^* \times \Sigma^*$.

$R = \{abb \to bba, bbb \to aaa\}$ over $\Sigma = \{a, b\}$.

*rewrite relation*: apply rule in context. $u \to_R v \iff$
$\exists x, y \in \Sigma^*, (l \to r) \in R :$
$u = xly \wedge xry = v$

*derivation* $u_0 \to_R u_1 \to_R \ldots$

example derivation

```
abbbb
---
bbabb
  ---
bbbba
 ---
baaaa
```

# Height Annotations (I)

*change heights*:
for each position $p$ count number of rule applications that contain $p$

Bala Ravikumar: Peg-solitaire, string rewriting systems and finite automata, TCS 2004

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b2 b1 a1
   -- -- --
b1 a2 a3 a2 a1
```

bounded change heights imply:

- termination
- effective preservation of regularity

# Height Annotations (II)

*match heights*:
for each rule application:
each height in reduct
is $1+$ minimal height in redex
Geser, Hofbauer, Waldmann: Match-Bounded String Rewriting Systems, AAECC 2004

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b1 b1 a1
   -- -- --
b1 a2 a2 a2 a1
```

bounded match heights imply:

- termination
- effective preservation of regularity

# Height Annotations (III)

*inverse match heights*: match-heights for the inverse derivation

Geser, Hofbauer, Waldmann: Termination Proofs for String Rewriting Systems via Inverse Match-Bounds, JAR 2005.

```
a1 b1 b1 b1 b1
-- -- --
b0 b1 a1 b1 b1
      -- -- --
b0 b1 b1 b1 a0
   -- -- --
b0 a0 a0 a0 a0
```

bounded inverse match heights imply:

- termination is decidable
- effective preservation of CF

# Central Question

relation between (existence and then value) of bounds for match, change, inverse match heights?

match

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b1 b1 a1
   -- -- --
b1 a2 a2 a2 a1
```

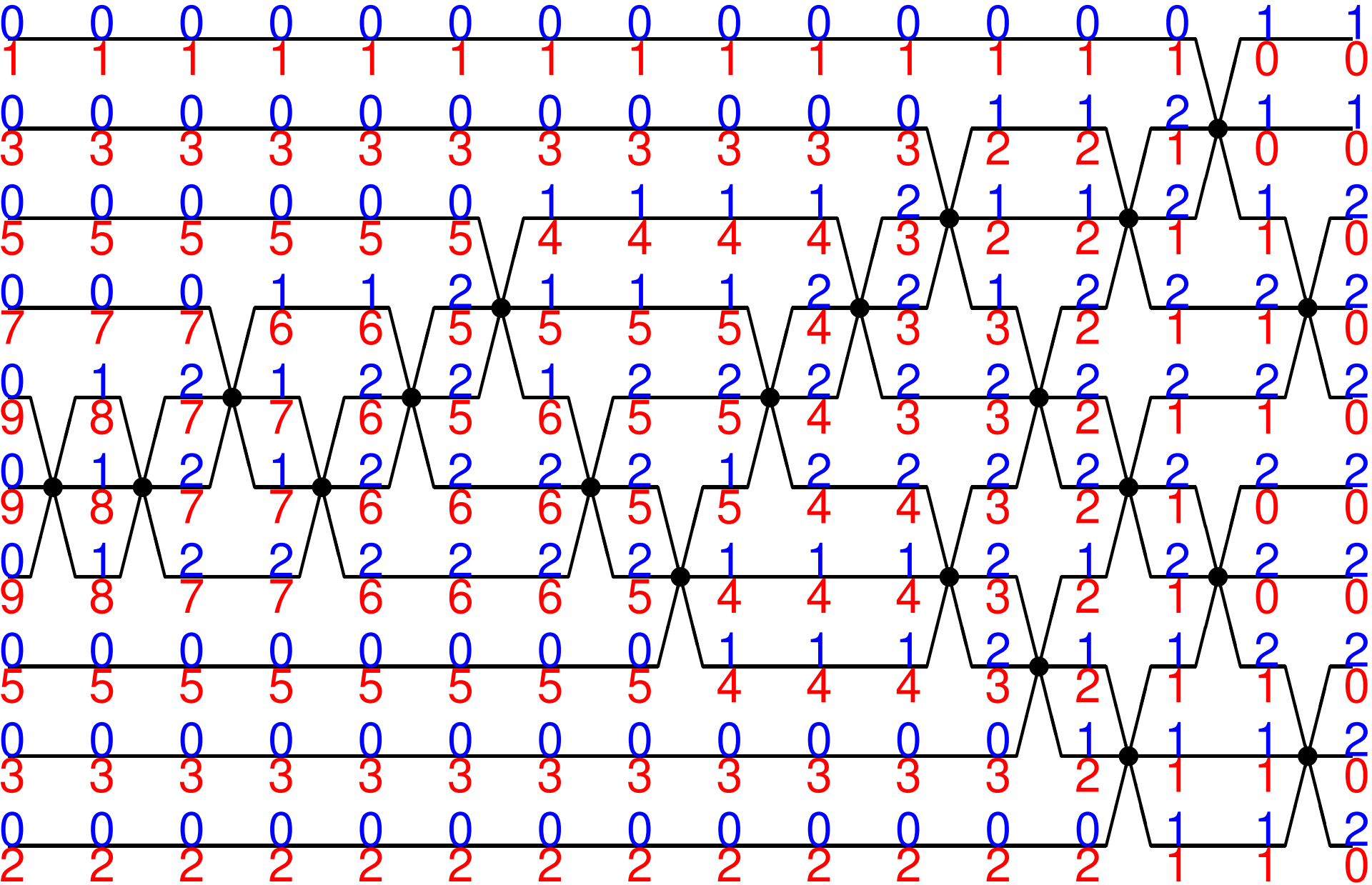
change

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b2 b1 a1
   -- -- --
b1 a2 a3 a2 a1
```

inverse match

```
a1 b1 b1 b1 b1
-- -- --
b0 b1 a1 b1 b1
      -- -- --
b0 b1 b1 b1 a0
   -- -- --
b0 a0 a0 a0 a0
```

# Example ($w = 3, M = 2, M' = 9$)



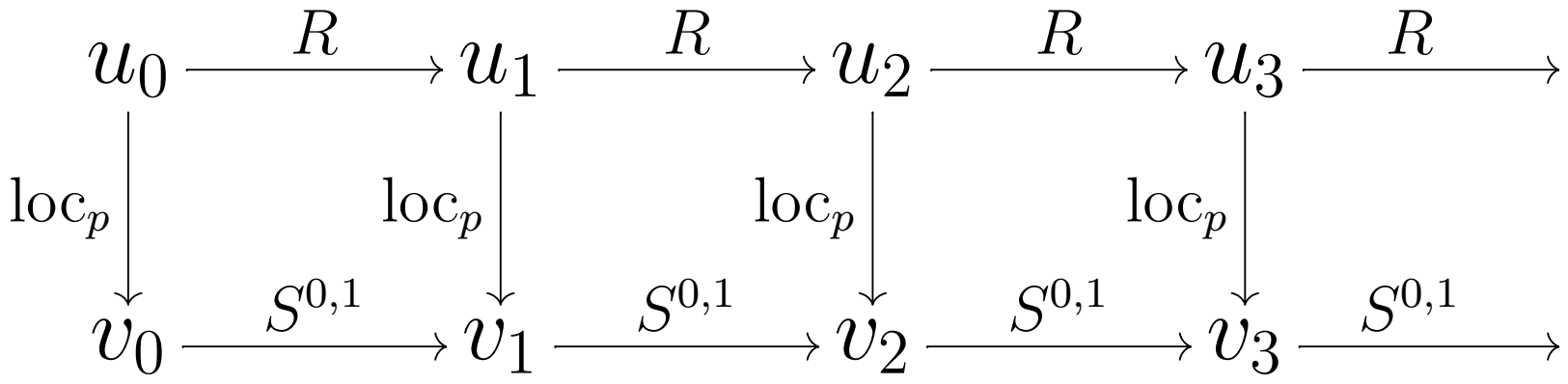
# Answer

for length-preserving systems $R$,
the following are equivalent:

1. $R$ is match-bounded
2. $R$ is change-bounded
3. $R$ is inverse change-bounded
4. $R$ is inverse match-bounded

Note: only need to show $1 \Rightarrow 2$, since $1 \Leftarrow 2$ is easy, and $2 \Rightarrow 3$ is trivial, rest follows by symmetry.
NB: This will *not* give a sharp bound for $1 \Rightarrow 4$.

# Proof Sketch ($1 \Rightarrow 2$)




fix a position $p$, use *localization* (down arrows):

- if $R$ step touches $p$, then local step $S^1$ (else $S^{0,1}$).
- $|v_0|$ is bounded (indep. of $|u_0|$)
- if top row is match-bounded, then bottom is mb.

$\Rightarrow$ number of $S$ steps is bounded,

$\Rightarrow$ number of $R$ steps that touch $p$ is bounded,

$\Rightarrow$ $R$ is change-bounded at $p$.

# Tool: Min/+ Algebra

the *tropical* semi-ring on $\mathbb{N} \cup \{\infty\}$:

- addition $\oplus$ is $\min$, zero: $\infty$
- multiplication $\otimes$ is $+$, unit: $0$

action of rewrite step on match heights
is a min/+ linear operator (matrix):

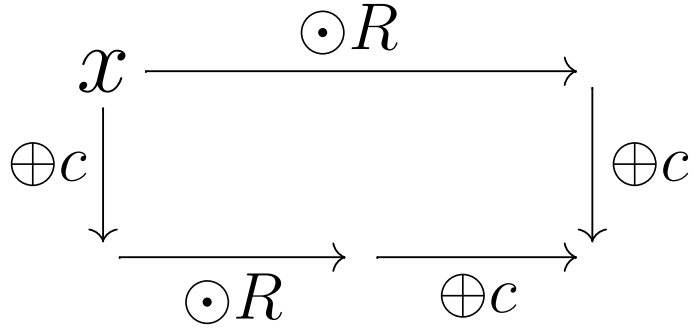
$$(1, 0, \underline{6, 2}, 5) \odot \begin{pmatrix} 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty \\ \infty & \infty & 1 & 1 & \infty \\ \infty & \infty & 1 & 1 & \infty \\ \infty & \infty & \infty & \infty & 0 \end{pmatrix} = (1, 0, \underline{3, 3}, 5)$$

cf: Tetris is max/+ linear, Gaubert et al, STACS 97

# Localization (I)

$\mathrm{loc}_p(x) := x \oplus c$ point-wise
for $c = 1^{w-1}2^{w-1} \ldots M^{2w-1} \ldots 2^{w-1}1^{w-1}$
centered at position $p$ and cutting off
for any step: $c \odot R \geq c$, thus $c \odot R \oplus c = c$.

This diagram commutes for any $x$:

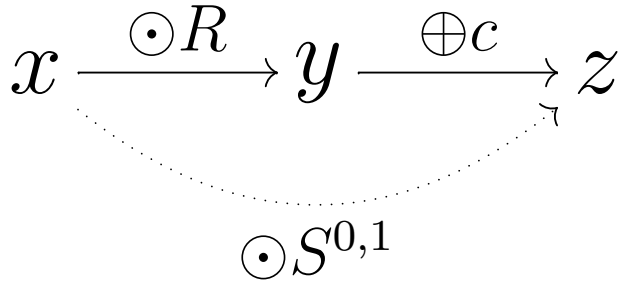



$$x \odot R \oplus c = x \odot R \oplus c \odot R \oplus c = (x \oplus c) \odot R \oplus c.$$

NB: if $p \in R$ (rewrite step that touches $p$),
then $x \neq c$. (uses match-boundedness of $R$-steps)

# Localization (II)

For any $x \leq c$ and rewrite step $R$, there is some (possibly empty) rewrite step $S$ such that:




Example: $w = 3, c = (\ldots, 2, 3, 3, 4, 4, \ldots),$

$x = (\ldots, 0, 0, \underline{3, 3, 4}, \ldots),$

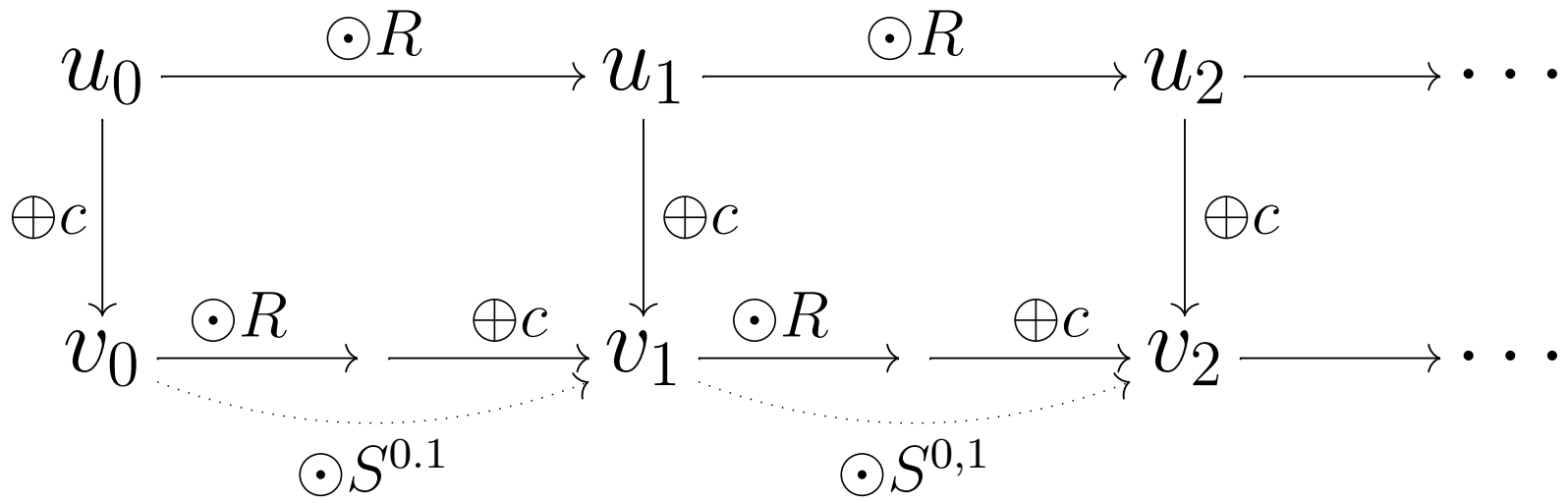
$y = x \odot R = (\ldots, 0, 0, 4, 4, 4, \ldots),$

$z = y \oplus c = (\ldots, 0, 0, 3, 4, 4, \ldots).$

The $S$-Redex in $x$ is $(\ldots, 0, 0, 3, \underline{3, 4}, \ldots).$

Proof: if $x =_{\mathrm{dom}\, R} c$, then $S$ empty, else $\mathrm{dom}\, S = \{q \in \mathrm{dom}\, R : y(q) \leq c(q)\}$.

# Localization (III)




- $R$-derivation is match-bounded

$\Rightarrow$ $S$-derivation is match-bounded

- $|v_0|$ is fixed

$\Rightarrow$ length of $S$-derivation is bounded

$\Rightarrow$ $R$-derivation is change-bounded

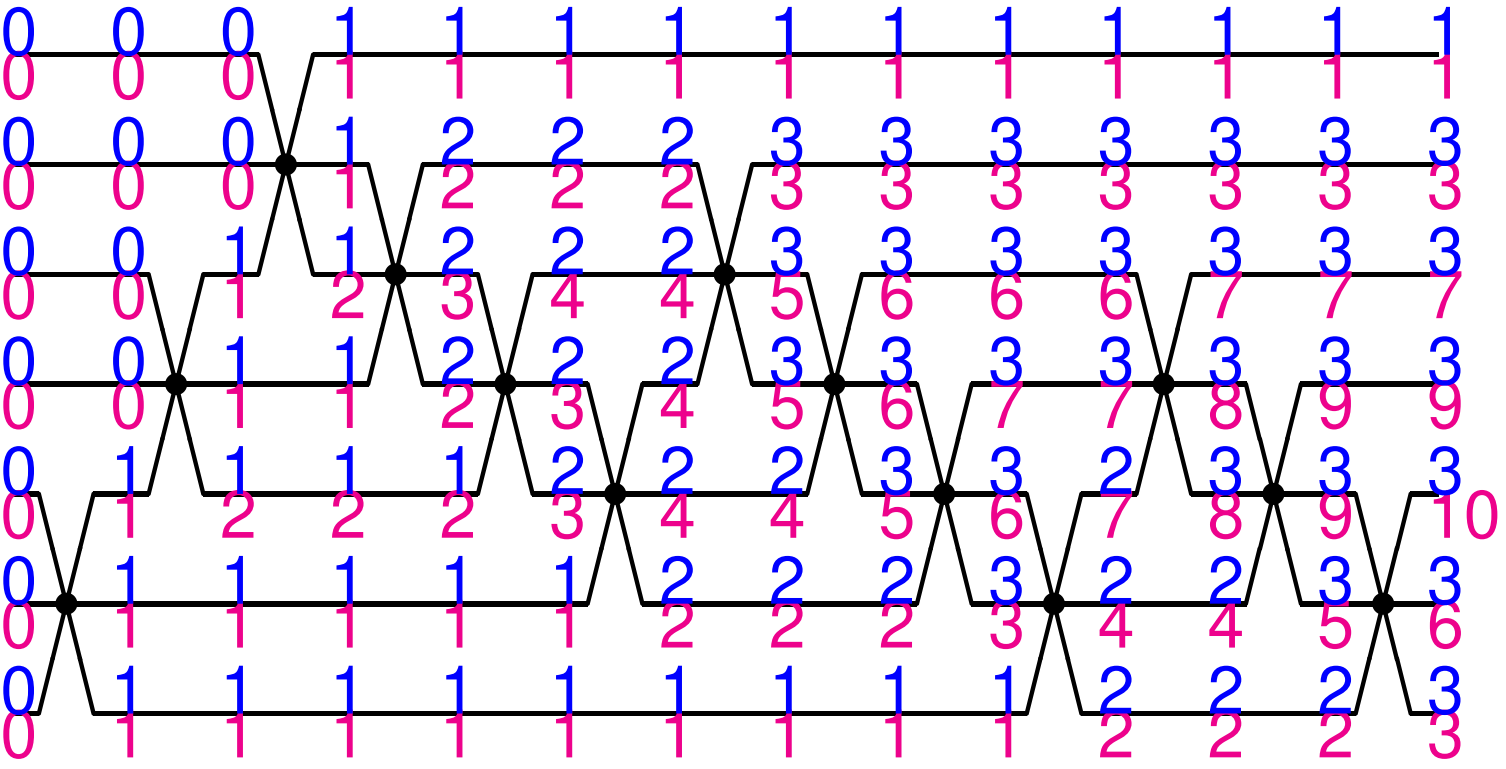
# Bounds from the Proof

match-bounded ($S$) derivation on string of length $l$ has $\leq l(w+1)^M$ steps

- original system $R$ is change-bounded by $2wM(w+1)^M$.
  (this bound seems attainable)
- this also bounds inverse match-heights
  (but this bound seems to be far off)

on next slides: families of derivations
with largest known growths

# Large Change Heights (I)




$$D_w(0) = []$$

$$D_w(M+1) = [0]\cdot \uparrow_M (D_w(M)) \cdot \ldots \cdot \uparrow_0 (D_w(M))$$

e.g. $D_3(3) = [0,\ 2,4,3,2,\ 1,3,2,1,\ 0,2,1,0]$

# Large Change Heights (II)

$$\begin{aligned} D_w(0) &= [] \\ D_w(M+1) &= [0]\cdot \uparrow_M (D_w(M)) \cdot \ldots \cdot \uparrow_0 (D_w(M)) \end{aligned}$$

- match-bounded by $M$
- length of derivation: $|D_w(M)| = \sum_{i=0}^{M} w^i$
- width of start string: $w \cdot M$
- by pigeonhole principle, there is position with change height $\geq w^M/(wM)$.
- cf. with upper bound from proof: $2wM(w+1)^M$

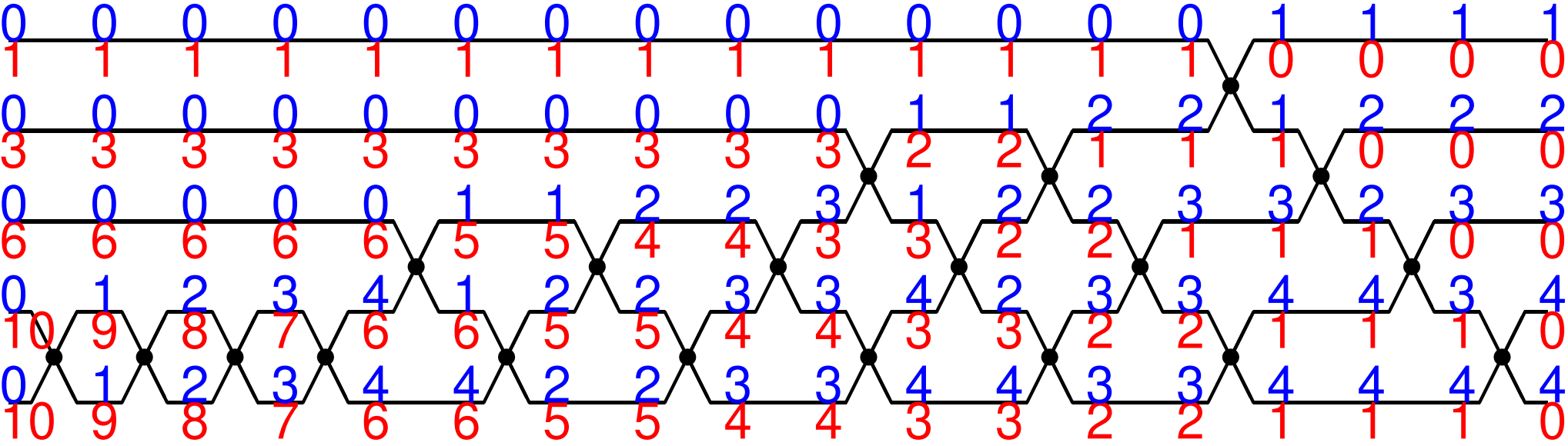
# Large Inverse Match Heights (I)

given width of rule $w$ , match-bound $M$
maximize inverse match-bound $M'$.

| $M'$ | $w=1$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $M=1$ | 1 | 2 | 2 | 2 | 2 |
| 2 | 2 | 6 | 9 | 12 | 15 |
| 3 | 3 | 12 | 20 | | |
| 4 | 4 | 19 | | | |
| 5 | 5 | 26 | | | |

found by computer search

# Large Inverse Match Heights (II)




$$E(2,M) = B(1,M) \cdot B(2,M-1) \cdot \ldots \cdot B(M,1)$$

$$B(v,h) = [v, v-1, \ldots, 1]^h$$

match-bound: $M$, inv. mb: $M + \ldots + 1 = \Theta(M^2)$.

in general, inverse match-bound is $\Theta(wM^2)$.

# Conclusion, Discussion

Conclusion

- min/plus algebra is a useful tool
  in match-bounded rewriting

Open Questions

- exact upper bounds for inverse match heights
  (polynomial $wM^2$ or exponential $w^M$ ?)
- drop the length-preserving restriction
  (but then cannot use proof via change heights)