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On multiplier modules of Hilbert C*-modules

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There are at least ***two approaches*** to a theory of multiplier modules of Hilbert C*-modules.

I would like to stay with the approach by Damir Bakić and Boris Guljaš (2003) in [5,6] which focusses on a given C*-algebra A and on a given full Hilbert A-module $\{X, \langle .,. \rangle\}$ where the A-valued inner product $\langle .,. \rangle$ on X is considered within its class of unitary equivalence.

This allows to rely on the notions of "compact" and adjointable operators without reference to possibly existing not unitary equivalent inner products on X inducing equivalent norms. $K_A(X)$ becomes unique.

Noticed by Bartosz Kwasniewski I became aware of the double-centralizer based definition of multiplier modules initiated by Matthew Daws in [D1,D2] (2010) and considered in depth by Alcides Buss, Bartosz Kwasniewski, Andrew McKee and Adam Skalski in [BKMS] (2024). Cf. [EchRae, Delfín] for representations.

They work with Banach A-B bimodules X connecting two given C*-algebras A and B as *-correspondences, specializing finally on imprimitivity bimodules (or: Hilbert A-B bimodules) in the style of strong Morita equivalence of C*-algebras.

C*-valued inner products appear late in the considerations.

Theorem:

[Lance, Thm.], [Blecher, Thms. 3.1, 3.2], [Frank, Thm. 5], [Solel, Thm. 1.1]

Let A be a C*algebra and X be a left Banach A-module the norm of which is known to be generated by an A-valued inner product on X with unknown values.

Then this A-valued inner product <.,.> on X is unique, and the values can be recovered by the formulae

$$\langle x,x\rangle := \sup\{r(x)^* r(x) : r \text{ in } X' \text{ with } \|r\| \leq 1\}$$

$$\langle x,y\rangle := \frac{1}{4} \cdot \sum_{k=1..4} \mathbf{i}^k \cdot \langle x+\mathbf{i}^k y, x+\mathbf{i}^k y\rangle$$

for every x,y in X , where the right side uses the norm of the underlying Banach A-module only.

Consequences:

Every bijective isometric A-linear isomorphism of two Hilbert A-modules T identifies the two A-valued inner products by the formula

$\langle .,. \rangle_2 = \langle T(.), T(.) \rangle_1$, $T^* = T^{-1}$ if onto, and vice versa.

[Solel, Thm. 3.3]

The Linking C*-algebras of two isometrically isomorphic Hilbert A-modules are *-isomorphic.

The completely boundedness structures on isometrically isomorphic Hilbert A-modules are the same, derived from the A-valued inner product (values).

Settings …

C*-algebras:

Norm-closed *-subalgebras of W*-algebras B(H) of all bounded linear operators on Hilbert spaces H .

W*-algebras or von Neumann algebras:

Norm-closed *-subalgebras of W*-algebras B(H) of all bounded linear operators on Hilbert spaces H which coincide with their bicommutant.

We focus on **non-unital C*-algebras A** to get meaningful derived constructions …

… and on C*-algebras where they are **two-sided ideals**.

C*-algebras

Considering C*-algebras (and Hilbert C*-modules) there are some points to respect:

- Possible existence of non-zero zero-divisors
- Possible non-commutativity of multiplication
- Inner one-/two-sided norm-closed ideal structure
- Possible existence of non-trivial orthogonal and skew projections
- … a rich algebraic, topological, noncommutative theory with more derived structures …

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Hilbert C*-modules (full – most offen today)

Let A be a C*-algebra and X be a complex-linear space and (right) A-module where both the complex-linear structures are compatible.

Let X admit a map <.,.>: $X \times X \to A$ conjugate-A-linear in the first and A-linear in the second variable such that

(i) $\langle x,y\rangle = \langle y,x\rangle^*$ for any x,y in M,

(ii) $0 \le \langle x,x\rangle$ for any x in M,

(iii) $\langle x,x\rangle = 0$ iff $x=0$.

Then $\|x\| := \|\langle x,x\rangle\|_A^{1/2}$ defines an A-module norm on X.

Examples .

Hilbert C*-modules

A Hilbert C*-module X over a C*-algebra A is said to be *full* iff A is the norm-closed linear span of the set { <x,y> : x,y in X } . We write A=<X,X> .

Two A-valued inner products on X are *unitarily equivalent* iff there exists a bounded *adjointable* invertible module operator T on X such that $<x,y>_2=<T(x),T(y)>_1$ for any x,y in X.

Multipliers of C*-algebras

A - a (non-unital) C*-algebra, faithfully *-represented on H

M(A) = {m in B(H): ma, am in A for any a in A}
- a unital C*-algebra where A is a two-sided norm-closed ideal (the multiplier algebra of A)

LM(A) = RM(A)* = {m in B(H) : ma in A for any a in A}
- a unital Banach algebra (the left/right multiplier algebra)

QM(A) = {m in B(H) : bma in A for any a,b in A}
- an involutive Banach space (the quasi-multiplier space)

Multiplier algebras

Universal property of M(A):

For any C*-algebra *D* containing *A* as a two-sided ideal, there exists a unique *-homomorphism $\varphi: D \to M(A)$ such that φ extends the identity homomorphism on *A* and $\varphi(A^{\perp}) = \{0\}$.

In particular, if the two-sided ideal *A* is an essential ideal in D (i.e. $A^{\perp}=\{0\}$ in D) then D is *-isometrically embeddable in M(A).

So, M(A) is the largest C*-algebra where A is essential.

Multipliers of C*-algebras

The elements of the different types of multipliers can be calculated w.r.t. any von Neumann or monotone complete C*-algebra where A is faithfully *-represented, cf. [7,11,28,32]. We get C*-isomorphisms.

Multiplier algebras might admit an entire lattice of non-unital, two-sided, non-isomorphic ideals A_α such that $M(A_\alpha) = M(A_\beta)$ for any two of them, cf. [20].

There are other calculation methods like double centralizers …

Operators on Hilbert C*-modules cf. [28]

$K_A(X)$ - the C*-algebra of all "compact" A-linear operators on X, norm-closed linear hull of elementary operators $\{\theta_{x,y} : \theta_{x,y} (z) = y \cdot <x,z>$ with z in $X\}$

$End_A{}^*(X)$ - the C*-algebra of all bounded adjointable A-linear operators on X $= M(K_A(X))$

$End_A(X)$ - the Banach algebra of all bounded A-linear operators on X $= LM(K_A(X))$

$End_A(X,X')$ - the Banach space of all bounded A-linear operators from X to X‘ $= QM(K_A(X))$

Example

$$A = \begin{pmatrix} c & c_0 \\ c_0 & c_0 \end{pmatrix}$$

$$M(A) = \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix}, \; LM(A) = RM(A)^* = \begin{pmatrix} c & l_\infty \\ c_0 & l_\infty \end{pmatrix},$$

$$QM(A) = \begin{pmatrix} c & l_\infty \\ l_\infty & l_\infty \end{pmatrix}.$$

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Example (continued)

Consider A as a Hilbert A-module over itself in two ways:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in LM(A) \setminus M(A)$$

$\langle a,b \rangle_A := a^*b$ for a,b in A

$\langle a,b \rangle_1 := a^*T^*Tb = \langle T(a),T(b) \rangle_A$ for a,b in A

One can calculate:

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix} \right\rangle_A = \begin{pmatrix} c & c \\ c & l_\infty \end{pmatrix} \notin M(A) \,.$$

Example

Then:

$A = K_A(A) \neq K_{A,1}(A)$ *-isomorphically as C*-algebras

$M(A) = End_A^*(A) \neq End_{A,1}^*(A)$ *-isomorphically as C*-algebras

cf. [11].

Either M(A)=LM(A) and LM(A)=QM(A),
or $M(A) \subset LM(A) \subset QM(A)$, cf. [8, Cor. 4.18].

Multiplier modules of Hilbert C*-modules

Multiplier modules ([5,6])

Let {X, <.,.>} be a full Hilbert C*-module over a given (non-unital) C*-algebra A . An extension of X is a triple (Y,B,Φ) such that

(i) B is a C*-algebra containing A as a two-sided norm-closed ideal.

(ii) Y is a Hilbert B-module.

(iii) Φ : X → Y is a bounded module map satisfying $\langle \Phi(x), \Phi(y)\rangle_Y = \langle x,y\rangle_X$ for any x,y in X.

(iv) Im(Φ) = Y·A = {za : z in Y, a in A} = {x in Y : <x,x> in A}

The triple (Y,B,Φ) is an essential extension of X, if A is an essential ideal of B. Then M(X) is the maximal one.

Multiplier modules ([5,6])

Let $\{X, \langle .,. \rangle\}$ be a (not necessarily full) Hilbert C*-module over a given (non-unital) C*-algebra A. Denote by M(X) the set of all adjointable maps from A to X, i.e. $M(X) := End_A^*(A,X)$.

Obviously, M(X) is a Hilbert M(A)-module with the M(A)-valued inner product $\langle z_1,z_2 \rangle = z_1^* z_2$ for z_1,z_2 in M(X). The resulting Hilbert M(A)-module norm coincides with the operator norm on M(X).

We call M(X) the **multiplier module** of X. It is a maximal extension of X, in case X is full. $X = K_A(A,X)$.

Multiplier modules [EchRae,Schw,D1,D2,BKMS,Del]

Let A,B be strongly Morita equivalent C*-algebras and X be an appropriate imprimitivity A-B bimodule.

A pair of maps (R,L) in $End_A(A,X) \oplus End_B(B,X)$ such that

$$aL(b) = R(a)b, \quad \text{for any } a \text{ in } A,\ b \text{ in } B,$$

is called a multiplier of X. The multiplier module M(X) of X is the set of all multipliers of X, a M(A)-M(B) Hilbert bimodule.

$$K_A(A,X) = K_B(B,X) = X \subseteq M(X) = End_A{}^*(A,X) = End_B{}^*(B,X)$$

Theorem: [EchRae, Prop. A.1]

Let A be a C*-algebra and X be a full Hilbert A-module.

Then the multiplier module M(X) can be isometrically isomorphically identified with the right upper corner of the multiplier algebra M(L(X)) of the linking algebra L(X) .

However, the linking algebra L(M(X)) is smaller-equal to M(L(X)), generally speaking, and L(M(X)) might be non-unital.

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Pairings and their uniqueness

Proposition:

For a given pair of C*-algebras (A,M(A)) , let X_1, X_2 be two full Hilbert C*-modules over A such that their multiplier modules $M(X_1)$, $M(X_2)$ are isometrically isomorphic as Hilbert M(A)-modules. Then X_1 and X_2 are isometrically isomorphic as Hilbert A-modules to $M(X_1)A = M(X_2)A$ resp..

So, the pairings (X,M(X)) are bound to each other for given C*-algebras (A,M(A)) up to unitary equivalence.

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Pairings and their uniqueness

Proposition:

Suppose, we have two non-isomorphic C*-algebras A_1 and A_2 such that they admit the same multiplier C*-algebra $M(A)$. Let X_1 be a full Hilbert A_1-module and X_2 a full Hilbert A_2-module such that $M(X_1)$ and $M(X_2)$ are unitarily isomorphic as Hilbert $M(A)$-modules.

Then X_1 is not unitarily isomorphic to X_2 as a Hilbert $M(A)$-module ($M(X_1)A_1=X_1$, $M(X_2)A_2=X_2$).

Topological characterization

Let A be a C*-algebra and X be a Hilbert A-module. Let the strict topology on M(X) be induced jointly by the two families of semi-norms $\{z \rightarrow za : a \text{ in } A\}$ and $\{\|\langle z,x\rangle\| : x \text{ in } X, \|x\| \leq 1\}$ for z in M(X) . It is a locally convex topology.

The multiplier module M(X) turns out to be complete with respect to this strict topology, and M(X) is the strict completion of X , [5, Thm. 1.8, 1.9].

Moreover, the strict completion is an idempotent operation, i.e. $M_{M(A)}(M_A(X)) = M_A(X)$.

A Morita-like point of view

Theorem: cf. [EchRae, Prop. 1.3]

Let A be a C*-algebra and M(A) be its multiplier algebra. Let X be a full (right) Hilbert A-module and M(X) be its multiplier module, a full (right) Hilbert M(A)-module.

Then M(X) is also the full (left) multiplier module of the (left) Hilbert $K_A(X)$-module X with respect to the pairing of C*-algebras $K_A(X)$ and $M(K_A(X)) = End_A^*(X) = M(K_{M(A)}(M(X))) = End_{M(A)}^*(M(X))$, and vice versa.

There are non-unital, strongly Morita equivalent C*-algebras without multiplier imprimitivity bimodules.

-Isomorphism of key operator C-algebras

[5, Thm. 2.3].

For X,Y Hilbert A-modules each operator T in $\mathrm{End}_A^*(X,Y)$ has an extension T_M in $\mathrm{End}_{M(A)}^*(M(X),M(Y))$ with the same norm value obtained as the strict continuation of T . Therefore, it is uniquely determined.

Moreover, every operator in $\mathrm{End}_{M(A)}^*(M(X),M(Y))$ arises this way, i.e. the C*-algebras $\mathrm{End}_{M(A)}^*(M(X),M(Y))$ and $\mathrm{End}_A^*(X,Y)$ are *-isomorphic.

This covers also the non-full variant of the definition.

"Compact" operator C*-algebras

Theorem:

Let A be a C*-algebra with multiplier algebra M(A) . Let X be a full Hilbert A-module and M(X) be its full multiplier module.

The C*-algebra $K_A(X)$ of all "compact" operators on X admits a *-isomorphic embedding into the C*-algebra $K_{M(A)}(M(X))$ of all "compact" operators on M(X). $K_A(X)$ is smaller than $K_{M(A)}(M(X))$ if $X \neq M(X)$.

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Bounded module operator algebras

Theorem:

Let A be a C*-algebra with multiplier algebra M(A) . Let X be a full Hilbert A-module and M(X) be its full multiplier module.

There does not exist any bounded M(A)-linear map $T_0 : M(X) \rightarrow M(X)$ such that $T_0 \neq 0$ on M(X), but $T_0 = 0$ on $X \subset M(X)$.

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Bounded module operator algebras

Theorem (continued):

The Banach algebra $\mathrm{End}_{M(A)}(M(X))$ admits an isometric embedding into the Banach algebra $\mathrm{End}_A(X)$ by restricting an element on the domain from $M(X)$ to $X \subset M(X)$.

If the left multiplier algebra of $K_A(X)$ is larger than the multiplier algebra of it, then $\mathrm{End}_{M(A)}(M(X))$ can be smaller than $\mathrm{End}_A(X)$, i.e. not every bounded module operator might admit a bounded module operator continuation. (Existing continuations are unique.)

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Bounded module functionals

Theorem:

Let A be a C*-algebra with multiplier algebra $M(A)$. Let X be a full Hilbert A-module and $M(X)$ be its full multiplier module.

There does not exist any bounded $M(A)$-linear map $f_0 : M(X) \rightarrow M(A)$ such that $f_0 \neq 0$ on $M(X)$, but $f_0 = 0$ on $X \subset M(X)$.

The Banach $M(A)$-module $M(X)'_{M(A)}$ admits an isometric modular embedding into the Banach A-module X'_A by restricting an element on the domain from $M(X)$ to $X \subset M(X)$. There exist examples such that X'_A is strictly larger than the embedded copy of $M(X)'_{M(A)}$.

Modular maps from X to X′

Theorem:

The same suppositions …

There does not exist any bounded M(A)-linear map $T_0 : M(X) \to M(X)'$ such that $T_0 \neq 0$ on $M(X)$, but $T_0 = 0$ on $X \subset M(X)$.

The Banach space $\mathrm{End}_{M(A)}(M(X),M(X)')$ admits an isometric embedding into the Banach space $\mathrm{End}_A(X,X')$ by restricting an element on the domain from $M(X)$ to $X \subset M(X)$. There exist examples such that $\mathrm{End}_A(X,X')$ is strictly larger than the embedded copy of $\mathrm{End}_{M(A)}(M(X),M(X)')$.

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Ideas and Questions

Are multiplier modules C*-reflexive as Hilbert C*-modules? (it depends on the choice of A or $M(A)$?)

Can certain C*-reflexive Hilbert C*-modules be characterized by intrinsic properties?

In general: $X \subseteq X'' \subseteq X'$, and all four pairs of inclusion relations appear in examples. $X''' = X'$.

Are there (interesting) families of pairwise non-*-isomorphic C*-algebras $K_A(X)$ based on (parametrized) families of non-adjointable invertible bounded module operators T on X changing the C*-valued inner product on X ?

Can two-sided norm-closed ideals A in M(A) and Hilbert A-modules X replaced by one-sided norm-closed ideals I in certain unital C*-algebras B to form „one-sided multiplier modules M(I)“ of them?

There exist first considerations by V. M. Manuilov, see arxiv.

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Discussions …

Thank you for your attention.

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Coda

Monotone complete C*-algebras:
C*-algebras for which each increasingly directed net of positive elements admits a supremum inside the algebra.

Generally true in C*-algebras:

$$xy^* = \frac{1}{4} \cdot \sum_{k=1}^{4} \mathbf{i}^k \cdot (x+\mathbf{i}^k y)(x+\mathbf{i}^k y)^* \quad \text{with} \quad \mathbf{i}^2 = -1,\ x,y \text{ in } A$$

Polarization identity.

For W*-algebras: There exists a **w*-topology** since they are dual Banach spaces. w*-completeness.

Order convergence of nets [M. Hamana, 1981, 1982]

A net $\{a_\alpha : \alpha \text{ in } I\}$ of elements of a monotone complete C*-algebra A converges in order to an element a in A iff there are bounded nets $\{a_\alpha^{(k)} : \alpha \text{ in } I\}$ and $\{b_\alpha^{(k)} : \alpha \text{ in } I\}$ of self-adjoint elements of A and elements $\{a^{(k)} : \alpha \text{ in } I\}$ in A with k=1,2,3,4 such that

(i) $0 \leq a_\alpha^{(k)} - a^{(k)} \leq b_\alpha^{(k)}$ for any k, any α

(ii) $\{b_\alpha^{(k)} : \alpha \text{ in } I\}$ is decreasingly directed and has greatest lower bound zero

(iii) $\sum_{k=1}^4 \mathbf{i}^k a_\alpha^{(k)} = a_\alpha$ for every α in I, $\sum_{k=1}^4 \mathbf{i}^k a^{(k)} = a$ where $\mathbf{i}^2 = -1$.

Independent of the choice of $\{a_\alpha^{(k)}\}$, $\{b_\alpha^{(k)}\}$, $a^{(k)}$.

H – a Hilbert space, A – a σ-unital C*-algebra

$A \odot H$ – the algebraic tensor product,

$$\langle a \odot h, b \odot g \rangle = a \langle h,g \rangle_H b^*$$

(a,b in A, h,g in H) becomes a pre-Hilbert A-module, with norm-completion $M = A \otimes H$.

$A^n \simeq A \otimes \mathbf{C}^n$ for any n in N.

$l_2(A) \simeq A \otimes l_2$, alternative description:

$$l_2(A) = \{ a=\{a_i\}_{i \text{ in } \mathbf{N}} : \Sigma_{j=1}^{\infty} a_j a_j^* \text{ converges w.r.t. } \|.\|_A \}$$

with inner product $\langle a,a \rangle = \Sigma_{j=1}^{\infty} a_j a_j^*$.

Theorem: (Kasparov, 1980) Stabilization Theorem

Every countably generated Hilbert A-module M over a σ-unital C*-algebra A possesses an embedding as an orthogonal summand of $l_2(A)$ in such a way that the orthogonal complement is isometrically isomorphic to $l_2(A)$ again, i.e. $M \oplus l_2(A) = l_2(A)$.

Theorem: (Serre-Swan, 1956, 1962; Kawamura, 2003)

Every algebraically finitely generated (Hilbert) C*-module M over a unital C*-algebra A is isomorphic to a (orthogonal) direct summand of a C*-module of type A^n for a finite number n , i.e. $M \oplus M^c = A^n$.

Theorems: (Paschke, 1973; Hamana, 1992)

Let M be a Hilbert A-module over a W*-algebra (resp., monotone complete C*-algebra) A . Denote ist A-dual Banach A-module by M' and its A-bidual Banach A-module by M'' .

Then the A-valued inner product continues from M to M' embedding $M \subseteq M' \equiv M''$ and thus, stabilizing the extension process by A-duality.

M' becomes a self-dual Hilbert A-module.

$End_A(M')$ is W* (resp. monotone complete).