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On multiplier modules of Hilbert C*-modules

Michael Frank

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Variation of an old German/Austrian saying:

„Die Tradition ist nicht die Anbetung der Asche, sondern die Weitergabe der Streichhölzer."

„The tradition is not the worship of ashes, but the passing on of matches."

Jan Josef Liefers (* 1964)
Schauspieler, Musiker, Regisseur, Produzent
Actor, Musician, Director, Producer

Motivations from groupoid theory …

Roughly speaking:

If G and H are Morita equivalent as groupoids, then $C^*_r(G)$ and $C^*_r(H)$ are strongly Morita equivalent C*-algebras,

as well as C*(G) and C*(H) are strongly Morita equivalent C*-algebras.

Morita equivalence of C*-algebras preserves quite a number of properties of both the C*-algebras. Some of them can be reinterpreted as groupoid properties.

There is no transfer to Morita equivalence bimodules of the respective (local) multiplier algebras!

Extending these ideas beyond the (co)actions of locally compact groups on C*-algebras to the actions of locally compact groupoids G with Haar systems on C*-algebras A one can associate groupoid crossed products, which are also C*-algebras.

Strong Morita equivalences of C*-algebras A, B with (co)actions of such groupoids give rise to $A \times_{\delta_A} G$-$B \times_{\delta_B} G$ imprimitivity bimodules for the (full, reduced) groupoid crossed product C*-algebras. [Ren1-4], [leGall], [BaajSkan], [EchRae], [KS], [vEW], ..

So their ideal structure can be studied and compared in detail.

Another variant are actions by inverse semigroups S and by associated locally compact r-discrete groupoids G_S on C*-algebras.

They induce several C*-algebra isomorphisms and strong Morita equivalences of derived crossed product C*-algebras.

Sometimes actions by certain Hilbert $M_{(loc)}(A)$-modules are considered related to injective *-isomorphisms of one C*-algebra A into $M_{(loc)}(B)$ of another C*-algebra B.

So the correlated ideal structures can be studied.

Here again come (local) multiplier modules into play.

[Ren3], [KS], [KM1-4], [BKMS], [AKM], [M], [Tay1-2] …

Let G and H denote second countable, locally compact, locally Hausdorff groupoids with Haar systems $\{\lambda^u\}_{g \text{ in } G^{(0)}}$ and $\{\beta^v\}_{v \text{ in } H^{(0)}}$, respectively.

Assume that $G^{(0)}$ and $H^{(0)}$ are Hausdorff.

Let S(X) be Connes's complex vector space of functions spanned by the elements of $C_c(U)$ for all open Hausdorff subsets U of X.

A locally Hausdorff locally compact space Z is a *G-space* if there is a continuous, open map $r_Z : Z \to G^{(0)}$ and a continuous map $(\gamma, z) \to \gamma \cdot z$ from

$$G * Z = \{ (\gamma, z) \text{ in } G \times Z : s_G(\gamma) = r_Z(z)\}$$

to Z such that $r_X(z) \cdot z = z$ for all z and $(\gamma \eta) \cdot z = = \gamma \cdot (\eta \cdot z)$ for all (γ, η) in $G \times G$ with $s_G(\eta) = r_Z(z)$.

The action is *free* if $\gamma \cdot z = z$ implies $\gamma = r(z)$ and *proper* if the map $\Theta : G * X \to X \times X$ given by $\Theta(\gamma, x) = (\gamma \cdot x, x)$ is a proper map of $G * Z$ into $Z \times Z$. Thus Θ is a closed map with the property that the inverse image of a compact set is compact.

Let G and H be locally Hausdorff locally compact groupoids. A (G, H)*-equivalence* is a locally Hausdorff locally compact space Z such that:

(i) Z is a free and proper left G-space;

(ii) Z is a free and proper right H-space:

(iii) the actions of G and H on Z commute;

(iv) r_Z induces a homeomorphism of Z/H onto $G^{(0)}$; and

(v) s_Z induces a homeomorphism of G\Z onto $H^{(0)}$.

If Z is a (G, H)-equivalence, then S(Z) is a S(G)-S(H)-bimodule with natural actions and pre-inner products.

S(Z) is a pre-S(G)-S(H)-imprimitivity bimodule with respect to the universal norms on S(G) and S(H) , and such that its completion X implements a **strong Morita equivalence between C*(G) and C*(H)** . (See [MRW], [MW], [SW].)

Similarly, but more technical:

There exists a $C^*_r(G)$-$C^*_r(H)$-imprimitivity bimodule isometrically isomorphic to $p_G \cdot C^*_r(L) \cdot p_H$, where L is the linking groupoid of G and H. Hence, **$C^*_r(G)$ and $C^*_r(H)$ are Morita equivalent**. (See [SW].)

See [Tu, AKM] for the non-Hausdorff case, in details.

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Settings …

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C*-algebras:

Norm-closed *-subalgebras of W*-algebras B(H) of all bounded linear operators on Hilbert spaces H .

W*-algebras or von Neumann algebras:

Norm-closed *-subalgebras of W*-algebras B(H) of all bounded linear operators on Hilbert spaces H which coincide with their bicommutant.

We focus on **non-unital C*-algebras A** to get meaningful derived constructions …

… and on C*-algebras where they are **two-sided ideals**.

C*-algebras

Considering C*-algebras (and Hilbert C*-modules) there are some points to respect:

- Possible existence of non-zero zero-divisors
- Possible non-commutativity of multiplication
- Inner one-/two-sided norm-closed ideal structure
- Possible existence of non-trivial orthogonal and skew projections
- … a rich algebraic, topological, noncommutative theory with more derived structures …

Hilbert C*-modules (full – most offen today)

Let A be a C*-algebra and X be a complex-linear space and (right) A-module where both the complex-linear structures are compatible.

Let X admit a map <.,.>: X×X → A conjugate-A-linear in the first and A-linear in the second variable such that

(i) $\langle x,y\rangle = \langle y,x\rangle^*$ for any x,y in M,

(ii) $0 \leq \langle x,x\rangle$ for any x in M,

(iii) $\langle x,x\rangle = 0$ iff $x=0$.

Then $\|x\| := \|\langle x,x\rangle\|_A^{1/2}$ defines an A-module norm on X.

Examples .

Hilbert C*-modules

A Hilbert C*-module X over a C*-algebra A is said to be *full* iff A is the norm-closed linear span of the set { $<x,y>$: x,y in X } . We write $A=<X,X>$.

Two A-valued inner products on X are *unitarily equivalent* iff there exists a bounded *adjointable* invertible module operator T on X such that $<x,y>_2=<T(x),T(y)>_1$ for any x,y in X.

Multipliers of C*-algebras

A - a (non-unital) C*-algebra, faithfully *-represented on H

M(A) = {m in B(H): ma, am in A for any a in A}
- a unital C*-algebra where A is a two-sided norm-closed ideal (the multiplier algebra of A)

LM(A) = RM(A)* = {m in B(H) : ma in A for any a in A}
- a unital Banach algebra (the left/right multiplier algebra)

QM(A) = {m in B(H) : bma in A for any a,b in A}
- an involutive Banach space (the quasi-multiplier space)

Multiplier algebras

Universal property of M(A):

For any C*-algebra D containing A as a two-sided ideal, there exists a unique *-homomorphism $\varphi: D \to M(A)$ such that φ extends the identity homomorphism on A and $\varphi(A^{\perp}) = \{0\}$.

In particular, if the two-sided ideal A is an essential ideal in D (i.e. $A^{\perp}=\{0\}$ in D) then D is *-isometrically embeddable in M(A).

So, M(A) is the largest C*-algebra where A is essential.

Multipliers of C*-algebras

The elements of the different types of multipliers can be calculated w.r.t. any von Neumann or monotone complete C*-algebra where A is faithfully *-represented, cf. [7,11,28,32]. We get C*-isomorphisms.

Multiplier algebras might admit an entire lattice of non-unital, two-sided, non-isomorphic ideals A_α such that $M(A_\alpha) = M(A_\beta)$ for any two of them, cf. [20].
(see Noetherian modules)

Beware – $M_{loc}(A)$ can be highly non-trivial.

Operators on Hilbert C*-modules cf. [28]

$K_A(X)$ - the C*-algebra of all "compact" A-linear operators on X, norm-closed linear hull of elementary operators $\{\theta_{x,y} : \theta_{x,y}(z) = y \cdot \langle x,z \rangle$ with z in $X\}$

$End_A^*(X)$ - the C*-algebra of all bounded adjointable A-linear operators on X $= M(K_A(X))$

$End_A(X)$ - the Banach algebra of all bounded A-linear operators on X $= LM(K_A(X))$

$End_A(X,X')$ - the Banach space of all bounded A-linear operators from X to X′ $= QM(K_A(X))$

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The double centralizer approach to multiplier modules

starts with two strongly Morita equivalent C*-algebras A, B and an A-B imprimitivity module X connecting them. ($B=K_A(X)$, $A=K_B(X)$.)

A *multiplier of X* is a pair $m=(m_A,m_B)$ of module maps, m_A: A → X, m_B: X → B such that

$m_A(a) \cdot b = a \cdot m_B(b)$ for any a in A, b in B.

Denotation of the multiplier module of X : M(X) .

X is isometrically A-B-bilinearly embeddable into M(X).

Facts (see [EchRae]):

M(X) is an A-B bimodule satisfying two conditions:

(i) $A \cdot M(X) \subseteq X$, $M(X) \cdot B \subseteq X$.

(ii) Any other A-B bimodule containing X and satisfying (i) admits an A-B bimodule homomorphism into M(X) acting as an identity map on X.

Two A-B bimodules satisfying (i)+(ii) are isomorphic to M(X) (uniqueness).

$\mathrm{End}_A{}^*(A,X) = M(X) = \mathrm{End}_B{}^*(X,B)$ $(m_A \leftarrow m \rightarrow m_B)$

(remember adjointability)

The Linking algebra picture:

$$L = \begin{pmatrix} A & X \\ \tilde{X} & B \end{pmatrix}$$

with operations coming from module actions and C*-valued inner products on X .

$$M(L(X)) = \begin{pmatrix} M(A) & M(X) \\ M(\tilde{X}) & M(B) \end{pmatrix}$$, but $L(M(X)) \subseteq M(L(X))$!

Nevertheless, the upper right corner of M(L(X)) is unitarily isomorphic to M(X) . And M(L(X)) might not be a linking algebra.

So M(X) can be a non-projective left or right module!

Example:

$A=\mathbb{C}$, $X=l_2$ – sep. Hilbert space, $B=K_{\mathbb{C}}(l_2)$

$M(X)=X$ – in this case, $L(M(X)) = \begin{pmatrix} \mathbb{C} & l_2 \\ l_2 & K_{\mathbb{C}}(l_2) \end{pmatrix} = L(X)$

$M(L(X)) = \begin{pmatrix} \mathbb{C} & l_2 \\ l_2 & B_{\mathbb{C}}(l_2) \end{pmatrix}$

Example

$$A = \begin{pmatrix} c & c_0 \\ c_0 & c_0 \end{pmatrix}$$

$$M(A) = \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix}, \; LM(A) = RM(A)^* = \begin{pmatrix} c & l_\infty \\ c_0 & l_\infty \end{pmatrix},$$

$$QM(A) = \begin{pmatrix} c & l_\infty \\ l_\infty & l_\infty \end{pmatrix}.$$

Example (continued)

Consider A as a Hilbert A-module over itself in two ways:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in LM(A) \setminus M(A)$$

$\langle a,b\rangle_A$:= a*b for a,b in A

$\langle a,b\rangle_1$:= a*T*Tb = $\langle T(a),T(b)\rangle_A$ for a,b in A

One can calculate:

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} c & c_0 \\ c_0 & l_\infty \end{pmatrix} \right\rangle_A = \begin{pmatrix} c & c \\ c & l_\infty \end{pmatrix} \not\in M(A) .$$

Example (continued)

Then:

$A = K_A(A) \neq K_{A,1}(A)$ *-isomorphically as C*-algebras

$M(A) = End_A^*(A) \neq End_{A,1}^*(A)$ *-isomorphically as C*-algebras

cf. [11].

Either M(A)=LM(A) and LM(A)=QM(A),

or $M(A) \subset LM(A) \subset QM(A)$, cf. [8, Cor. 4.18].

Multiplier modules of Hilbert C*-modules

There is a **second / third *approach*** to a theory of multiplier modules of Hilbert C*-modules.

I would like to stay with the approach by Damir Bakić and Boris Guljaš (2003) in [5,6] which focusses on a given C*-algebra A and on a given *full Hilbert A-module* {X, <.,.>} where the A-valued inner product <.,.> on X is considered within its class of *unitary equivalence*.

This allows to rely on the notions of "compact" and adjointable operators without reference to possibly existing not unitary equivalent inner products on X inducing equivalent norms. $K_A(X)$ becomes unique.

Multiplier modules ([5,6])

Let {X, <.,.>} be a full Hilbert C*-module over a given (non-unital) C*-algebra A . An extension of X is a triple (Y,B,Φ) such that

(i) B is a C*-algebra containing A as a two-sided norm-closed ideal.

(ii) Y is a Hilbert B-module.

(iii) Φ : X → Y is a bounded module map satisfying $\langle \Phi(x), \Phi(y)\rangle_Y = \langle x,y\rangle_X$ for any x,y in X.

(iv) Im(Φ) = Y·A = {za : z in Y, a in A} = {x in Y : <x,x> in A}

The triple (Y,B,Φ) is an essential extension of X, if A is an essential ideal of B. Then M(X) is the maximal one.

Multiplier modules ([5,6])

Let $\{X, \langle .,. \rangle\}$ be a (not necessarily full) Hilbert C*-module over a given (non-unital) C*-algebra A. Denote by M(X) the set of all adjointable maps from A to X, i.e. $M(X) := End_A{}^*(A,X)$.

Obviously, M(X) is a Hilbert M(A)-module with the M(A)-valued inner product $\langle z_1, z_2 \rangle = z_1{}^* z_2$ for z_1, z_2 in M(X). The resulting Hilbert M(A)-module norm coincides with the operator norm on M(X).

We call M(X) the **multiplier module** of X. It is a maximal extension of X, in case X is full. $X = K_A(A,X)$.

Pairings and their uniqueness (I)

Proposition:

For a given pair of C*-algebras (A,M(A)) , let X_1, X_2 be two full Hilbert C*-modules over A such that their multiplier modules $M(X_1)$, $M(X_2)$ are isometrically isomorphic as Hilbert M(A)-modules. Then X_1 and X_2 are isometrically isomorphic as Hilbert A-modules to $M(X_1)A = M(X_2)A$ resp..

So, the pairings (X,M(X)) are bound to each other for given C*-algebras (A,M(A)) up to unitary equivalence.

Pairings and their uniqueness (II)

Proposition:

Suppose, we have two non-isomorphic C*-algebras A_1 and A_2 such that they admit the same multiplier C*-algebra $M(A)$. Let X_1 be a full Hilbert A_1-module and X_2 a full Hilbert A_2-module such that $M(X_1)$ and $M(X_2)$ are unitarily isomorphic as Hilbert $M(A)$-modules.

Then X_1 is not unitarily isomorphic to X_2 as a Hilbert $M(A)$-module ($M(X_1)A_1=X_1$, $M(X_2)A_2=X_2$).

Topological characterization

Let A be a C*-algebra and X be a Hilbert A-module. Let a ("the") strict topology on M(X) be induced jointly by the two families of semi-norms {||za|| : a in A} and {||<z,x>|| : x in X, ||x||≤1} for z in M(X) . It is a locally convex topology.

The multiplier module M(X) turns out to be complete with respect to this strict topology, and M(X) is the strict completion of X , [5, Thm. 1.8, 1.9].

Moreover, the strict completion is an idempotent operation, i.e. $M_{M(A)}(M_A(X)) = M_A(X)$.

A Morita-like point of view

Theorem: cf. [EchRae, Prop. 1.3]

Let A be a C*-algebra and M(A) be its multiplier algebra. Let X be a full (right) Hilbert A-module and M(X) be its multiplier module, a full (right) Hilbert M(A)-module.

Then M(X) is also the full (left) multiplier module of the (left) Hilbert $K_A(X)$-module X with respect to the pairing of C*-algebras $K_A(X)$ and $M(K_A(X)) = End_A{}^*(X) = M(K_{M(A)}(M(X))) = End_{M(A)}{}^*(M(X))$, and vice versa.

There are non-unital, strongly Morita equivalent C*-algebras without multiplier imprimitivity bimodules.

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-Isomorphism of key operator C-algebras

[5, Thm. 2.3].

For X,Y Hilbert A-modules each operator T in $End_A^*(X,Y)$ has an extension T_M in $End_{M(A)}^*(M(X),M(Y))$ with the same norm value obtained as the strict continuation of T . Therefore, it is uniquely determined.

Moreover, every operator in $End_{M(A)}^*(M(X),M(Y))$ arises this way, i.e. the C*-algebras $End_{M(A)}^*(M(X),M(Y))$ and $End_A^*(X,Y)$ are *-isomorphic.

This covers also the non-full variant of the definition.

"Compact" operator C*-algebras

Theorem:

Let A be a C*-algebra with multiplier algebra M(A) . Let X be a full Hilbert A-module and M(X) be its full multiplier module.

The C*-algebra $K_A(X)$ of all "compact" operators on X admits a *-isomorphic embedding into the C*-algebra $K_{M(A)}(M(X))$ of all "compact" operators on M(X). $K_A(X)$ is smaller than $K_{M(A)}(M(X))$ if $X \neq M(X)$.

Bounded module operator algebras

Theorem:

Let A be a C*-algebra with multiplier algebra M(A) . Let X be a full Hilbert A-module and M(X) be its full multiplier module.

There does not exist any bounded M(A)-linear map $T_0 : M(X) \rightarrow M(X)$ such that $T_0 \neq 0$ on M(X), but $T_0 = 0$ on $X \subset M(X)$.

Bounded module operator algebras

Theorem (continued):

The Banach algebra $End_{M(A)}(M(X))$ admits an isometric embedding into the Banach algebra $End_A(X)$ by restricting an element on the domain from $M(X)$ to $X \subset M(X)$.

If the left multiplier algebra of $K_A(X)$ is larger than the multiplier algebra of it, then $End_{M(A)}(M(X))$ can be smaller than $End_A(X)$, i.e. not every bounded module operator might admit a bounded module operator continuation. (Existing continuations are unique.)

Bounded module functionals

Theorem:

Let A be a C*-algebra with multiplier algebra M(A). Let X be a full Hilbert A-module and M(X) be its full multiplier module.

There does not exist any bounded M(A)-linear map $f_0 : M(X) \rightarrow M(A)$ such that $f_0 \neq 0$ on M(X), but $f_0 = 0$ on $X \subset M(X)$.

The Banach M(A)-module $M(X)'_{M(A)}$ admits an isometric modular embedding into the Banach A-module X'_A by restricting an element on the domain from M(X) to $X \subset M(X)$. There exist examples such that X'_A is strictly larger than the embedded copy of $M(X)'_{M(A)}$.

Modular maps from X to X′

Theorem:

The same suppositions …

There does not exist any bounded M(A)-linear map $T_0 : M(X) \rightarrow M(X)'$ such that $T_0 \neq 0$ on $M(X)$, but $T_0 = 0$ on $X \subset M(X)$.

The Banach space $End_{M(A)}(M(X),M(X)')$ admits an isometric embedding into the Banach space $End_A(X,X')$ by restricting an element on the domain from $M(X)$ to $X \subset M(X)$. There exist examples such that $End_A(X,X')$ is strictly larger than the embedded copy of $End_{M(A)}(M(X),M(X)')$.

Q

Ideas and Questions

Are multiplier modules C*-reflexive as Hilbert C*-modules?
(it depends on the choice of A or M(A) ?)

Can certain C*-reflexive Hilbert C*-modules be characterized by intrinsic properties?

In general: $X \subseteq X'' \subseteq X'$, and all four pairs of inclusion relations appear in examples. $X''' = X'$.

Are there (interesting) families of pairwise non-*-isomorphic C*-algebras $K_A(X)$ based on (parametrized) families of non-adjointable invertible bounded module operators T on X changing the C*-valued inner product on X ?

Can two-sided norm-closed ideals A in M(A) and Hilbert A-modules X replaced by ***one-sided*** norm-closed ideals I in certain unital C*-algebras B to form „one-sided multiplier modules M(I)“ of them?

There exist first considerations by V. M. Manuilov, see arxiv.

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Discussions …

Thank you for your attention.

Coda

Noticed by Bartosz Kwasniewski I became aware of the double-centralizer based definition of multiplier modules initiated by Matthew Daws in [D1,D2] (2010) and considered in depth by Alcides Buss, Bartosz Kwasniewski, Andrew McKee and Adam Skalski in [BKMS] (2024). Cf. [EchRae, Delfín] for representations.

They work with Banach A-B bimodules X connecting two given C*-algebras A and B as *-correspondences, specializing finally on imprimitivity bimodules (or: Hilbert A-B bimodules) in the style of strong Morita equivalence of C*-algebras.

C*-valued inner products appear late in the considerations.

Theorem:

[Lance, Thm.], [Blecher, Thms. 3.1, 3.2], [Frank, Thm. 5], [Solel, Thm. 1.1]

Let A be a C*algebra and X be a left Banach A-module the norm of which is known to be generated by an A-valued inner product on X with unknown values.

Then this A-valued inner product <.,.> on X is unique, and the values can be recovered by the formulae

$$\langle x,x\rangle := \sup \{r(x)^* r(x) : r \text{ in } X' \text{ with } \|r\| \leq 1\}$$

$$\langle x,y\rangle := \tfrac{1}{4} \cdot \Sigma_{k=1..4}\, \mathbf{i}^k \cdot \langle x+\mathbf{i}^k y, x+\mathbf{i}^k y\rangle$$

for every x,y in X , where the right side uses the norm of the underlying Banach A-module only.

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Consequences:

Every bijective isometric A-linear isomorphism of two Hilbert A-modules T identifies the two A-valued inner products by the formula

$<.,.>_2 = <T(.), T(.)>_1$, $T^*=T^{-1}$ if onto, and vice versa.

[Solel, Thm. 3.3]

The Linking C*-algebras of two isometrically isomorphic Hilbert A-modules are *-isomorphic.

The completely boundedness structures on isometrically isomorphic Hilbert A-modules are the same, derived from the A-valued inner product (values).

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Multiplier modules [EchRae,Schw,D1,D2,BKMS,Del]

Let A,B be strongly Morita equivalent C*-algebras and X be an appropriate imprimitivity A-B bimodule.

A pair of maps (R,L) in $\mathrm{End}_A(A,X) \oplus \mathrm{End}_B(B,X)$ such that

$$aL(b)= R(a)b, \quad \text{for any } a \text{ in } A,\ b \text{ in } B,$$

is called a multiplier of X. The multiplier module M(X) of X is the set of all multipliers of X, a M(A)-M(B) Hilbert bimodule.

$$K_A(A,X) = K_B(B,X) = X \subseteq M(X) = \mathrm{End}_A{}^*(A,X) = \mathrm{End}_B{}^*(B,X)$$

Theorem: [EchRae, Prop. A.1]

Let A be a C*-algebra and X be a full Hilbert A-module.

Then the multiplier module M(X) can be isometrically isomorphically identified with the right upper corner of the multiplier algebra M(L(X)) of the linking algebra L(X) .

However, the linking algebra L(M(X)) is smaller-equal to M(L(X)), generally speaking, and L(M(X)) might be non-unital.

Monotone complete C*-algebras:

C*-algebras for which each increasingly directed net of positive elements admits a supremum inside the algebra.

Generally true in C*-algebras:

$$xy^* = \frac{1}{4} \cdot \sum_{k=1}^{4} \mathbf{i}^k \cdot (x+\mathbf{i}^k y)(x+\mathbf{i}^k y)^* \quad \text{with} \quad \mathbf{i}^2 = -1, \; x,y \text{ in } A$$

Polarization identity.

For W*-algebras: There exists a **w*-topology** since they are dual Banach spaces. w*-completeness.

Order convergence of nets [M. Hamana, 1981, 1982]

A net $\{a_\alpha: \alpha \text{ in } I\}$ of elements of a monotone complete C*-algebra A converges in order to an element a in A iff there are bounded nets $\{a_\alpha^{(k)}: \alpha \text{ in } I\}$ and $\{b_\alpha^{(k)}: \alpha \text{ in } I\}$ of self-adjoint elements of A and elements $\{a^{(k)} : \alpha \text{ in } I\}$ in A with k=1,2,3,4 such that

(i) $0 \leq a_\alpha^{(k)} - a^{(k)} \leq b_\alpha^{(k)}$ for any k, any α

(ii) $\{b_\alpha^{(k)}: \alpha \text{ in } I\}$ is decreasingly directed and has greatest lower bound zero

(iii) $\sum_{k=1}^4 \mathbf{i}^k a_\alpha^{(k)} = a_\alpha$ for every α in I, $\sum_{k=1}^4 \mathbf{i}^k a^{(k)} = a$ where $\mathbf{i}^2 = -1$.

Independent of the choice of $\{a_\alpha^{(k)}\}$, $\{b_\alpha^{(k)}\}$, $a^{(k)}$.

H – a Hilbert space, A – a σ-unital C*-algebra

$A \odot H$ – the algebraic tensor product,

$<a \odot h, b \odot g> = a <h,g>_H b^*$

(a,b in A, h,g in H) becomes a pre-Hilbert A-module, with norm-completion $M = A \otimes H$.

$A^n \simeq A \otimes \mathbf{C}^n$ for any n in N.

$l_2(A) \simeq A \otimes l_2$, alternative description:

$l_2(A) = \{ a=\{a_i\}_{i \text{ in } \mathbf{N}} : \Sigma_{j=1}^{\infty} a_j a_j^* \text{ converges w.r.t. } \|.\|_A \}$

with inner product $<a,a> = \Sigma_{j=1}^{\infty} a_j a_j^*$.

Theorem: (Kasparov, 1980) Stabilization Theorem

Every countably generated Hilbert A-module M over a σ-unital C*-algebra A possesses an embedding as an orthogonal summand of $l_2(A)$ in such a way that the orthogonal complement is isometrically isomorphic to $l_2(A)$ again, i.e. $M \oplus l_2(A) = l_2(A)$.

Theorem: (Serre-Swan, 1956, 1962; Kawamura, 2003)

Every algebraically finitely generated (Hilbert) C*-module M over a unital C*-algebra A is isomorphic to a (orthogonal) direct summand of a C*-module of type A^n for a finite number n , i.e. $M \oplus M^c = A^n$.

Theorems: (Paschke, 1973; Hamana, 1992)

Let M be a Hilbert A-module over a W*-algebra (resp., monotone complete C*-algebra) A . Denote ist A-dual Banach A-module by M′ and its A-bidual Banach A-module by M′′ .

Then the A-valued inner product continues from M to M′ embedding $M \subseteq M' \equiv M''$ and thus, stabilizing the extension process by A-duality.

M′ becomes a self-dual Hilbert A-module.

$End_A(M')$ is W* (resp. monotone complete).