Core matrix interpretations for proving termination of term rewrite systems

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Abstract

Matrix interpretations are powerful techniques for proving termination of term rewrite systems. Among them, the original paper that introduced the matrix interpretation technique, originally aimed at string rewriting, also described sets of matrices that inherently provide a well-founded relation and its required monotonic properties, simplifying proofs. To the extent of our knowledge, these special sets of matrices have not yet explicitly been used in a term rewriting setting. We report on the generalisation of these sets of matrices to a term rewriting setting. Several results have been formalised in Isabelle/HOL and are integrated in the CeTA certifier.

2012 ACM Subject Classification Theory of computation → Equational logic and rewriting; Theory of computation → Logic and verification

Keywords and phrases Term rewriting, core matrix interpretations, formalisation, Isabelle/HOL

Acknowledgements We want to thank Johannes Waldmann for providing us with a version of Matchbox that produces core matrix interpretation proofs, and we thank the WST reviewers for their helpful comments.

1 Introduction

In this paper we present the core matrix interpretations termination technique, originally designed for string rewrite systems (SRSs) [6] for integer matrices.

▶ **Example 1.** Consider the following SRS (TPDB Waldmann19/SRS_Standard/random-246).

$$baaa \to bbaa \qquad abbb \to bbba \qquad abbb \to bbaa$$

During termCOMP 2024, the tool Multum-non-Multa of Hofbauer [5] generated the following termination proof by a core matrix interpretation.

$$\alpha(a) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \alpha(b) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This proof could not be certified by CeTA [10] in year 2024.

We generalise the core matrix interpretation method to ordered rings so that in particular it can be used for matrices over rational numbers using δ-orders, similarly to [7, 8]. We further generalise the result from SRSs to term rewrite systems (TRSs). Both generalisations have been formalised using Isabelle/HOL [9] and they are fully integrated into the formalised rewriting framework IsaFoR/CeTA as of version 3.6. Consequently, proofs such as the one in Example 1 are now accepted by CeTA.

In addition to the aforementioned generalisations, during the formalisation process, a proof in [6] was found to be flawed. Whilst not required to deduce the end results of the termination technique, we provide a reconstructed proof nevertheless.

Our conditions on matrix interpretations are incomparable to other variants of monotone matrix interpretations from the literature [4, 3, 8]. Still, we show that every matrix interpretation of [4, 8] can be transformed into a core matrix interpretation. Moreover, the integration of core matrix interpretations has already successfully been tested: the Matchbox tool of Waldmann [11] generates certificates for termination proofs using core matrix interpretations, and all proofs are certified with CeTA version 3.6.

2 Preliminaries

We assume familiarity with term rewriting [1] and string rewriting and recall important notions and notations. $\mathcal{T}(\mathcal{F},\mathcal{V})$ will be the set of first-order terms that can be constructed from function symbols in $\mathcal{F}$ and variables in $\mathcal{V}$. Then, a term rewriting system (abbreviated TRS) is a binary relation in $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$. Given a rule in a TRS $\mathcal{R}$ of the form $\ell \to r$, a rewriting operation, denoted $\to_{\mathcal{R}}$, using such a rule consists of rewriting $C[\ell\sigma]$ to $C[r\sigma]$, where C is a term with exactly one special variable $\square$ that will be replaced with $\ell\sigma$ (resp. $r\sigma$), and where $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$ is a substitution. A string rewriting system (abbreviated SRS) is a binary relation in $\Sigma^* \times \Sigma^*$ given an alphabet Σ. Rewriting using a rule $\ell \to r$ in an SRS $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$ and consists in rewriting $x\ell y$ to xry for any $x, y \in \Sigma^*$. Termination of a TRS/SRS $\mathcal{R}$ is denoted by $SN(\to_{\mathcal{R}})$ and refers to the non-existence of an infinite sequence of rewrite operations. Termination of a TRS (resp. SRS) $\mathcal{R}$ *relative* to a TRS (resp. SRS) $\mathcal{S}$ is the property $SN(\to_{\mathcal{S}}{}^* \circ \to_{\mathcal{R}} \circ \to_{\mathcal{S}}{}^*)$ and it is abbreviated by $SN(\mathcal{R}/\mathcal{S})$.

Throughout this paper, only square matrices of size n (often left implicit) are considered. Then, we need a few notations: The zero matrix shall be denoted by $\mathbf{0}$, whilst the identity matrix shall be denoted by $\mathbf{1}$; The usual notation $A_{i,j}$ will be used to denote the component of A at the i-th row and j-th column; The non-strict comparison on matrices $\geq$ shall correspond to a weak decrease in coefficients component-wise; On the other hand, the strict comparison on matrices $>$ shall correspond to a weak decrease component-wise *in addition* to a strict decrease component-wise for at least one coefficient in the matrices; Then, N shall refer to the set of non-negative matrices, i.e. $\{A \mid A \geq \mathbf{0}\}$, while P shall refer to the set of non-negative and non-null matrices, i.e. $\{A \mid A > \mathbf{0}\}$; Moreover, given two sets of matrices S and T, the set ST shall denote the set $\{A_1 \times A_2 \mid A_1 \in S, A_2 \in T\}$; Following that, the set T^k shall refer to $TT \ldots T$ k-times, T^* shall denote the set $\bigcup_k T^k$ and T^0 shall be $\{\mathbf{1}\}$; Finally, the notation ST shall be extended to individual matrices, i.e. given a matrix d, the notation dT shall denote $\{d \times A \mid A \in T\}$.

3 Core matrix interpretations

In this section, we present the "core" matrix interpretation technique in an SRS setting, as it is described in [6], before moving to its generalisation from integers to further ordered rings and before moving to a TRS setting. Roughly speaking, the termination technique consists in turning the string rewrite setting into a matrix setting, where every possible rewrite operation would be mapped to a well-founded relation on integer matrices, such as the strict comparison on N, ensuring termination.

Concretely, if we consider the SRS $\mathcal{R}$, if we call α the interpretation of strings (i.e. the map from strings to matrices), if $\alpha(s) \in N$ holds for any string s, and if $\alpha(s) > \alpha(t)$ holds for any strings $s \to_{\mathcal{R}} t$, then $\mathcal{R}$ is strongly-normalising. To achieve relative termination, e.g. $SN(\to_{\mathcal{S}}{}^* \circ \to_{\mathcal{R}} \circ \to_{\mathcal{S}}{}^*)$, in addition to the previous criteria, satisfying the non-strict comparison on N for $\mathcal{S}$-rewrite operations is sufficient, i.e. $\alpha(s) \geq \alpha(t)$ for any strings $s \to_{\mathcal{S}} t$. This criterion is sufficient since $\geq^* \circ > \circ \geq^* \subseteq >$, meaning $\to_{\mathcal{S}}{}^* \circ \to_{\mathcal{R}} \circ \to_{\mathcal{S}}{}^*$ would map to $>$ which terminates.

Since our goal is first and foremost to implement this termination technique into IsaFoR/CeTA to be able to certify proofs by "core" matrix interpretations, the process of verifying the required criteria (namely $\forall s, t,\ s \to_{\mathcal{R}} t \implies \alpha(s) > \alpha(t); s \to_{\mathcal{S}} t \implies \alpha(s) \geq \alpha(t)$ in our case; let us call this criteria **Property 1**) *must* be computable. However, this is obviously not the case for **Property 1**, due to the quantification over all possible strings. Thus, the "core" matrix interpretation approach aims to reformulate this criteria in such a way that it becomes computable. Since s has been rewritten to t in **Property 1**, s can be split into a left context, the place where a rewrite rule was applied, and a right context, making **Property 1** equivalent to

$$\left.\begin{array}{lll}\forall x, y, \ell, r,\ (\ell \to r) & \in \mathcal{R} & \implies \alpha(x\ell y) > \alpha(xry) \\ \forall x, y, \ell, r,\ (\ell \to r) & \in \mathcal{S} & \implies \alpha(x\ell y) \geq \alpha(xry)\end{array}\right\} \textbf{Property 2}$$

While **Property 2** is easily lifted to TRSs by simply changing the domain and the definitions of left and right context, to further reformulate it we need to take into account SRS-specific properties. In particular, in [6], the interpretation of the string ab is defined as $\alpha(a) \times \alpha(b)$ considering $\{a, b\} \subseteq \Sigma$. Thus, **Property 2** can be reformulated as

$$\left.\begin{array}{lll}\forall x, y, \ell, r,\ (\ell \to r) & \in \mathcal{R} & \implies \alpha(x)(\alpha(\ell) - \alpha(r))\alpha(y) > \mathbf{0} \\ \forall x, y, \ell, r,\ (\ell \to r) & \in \mathcal{S} & \implies \alpha(x)(\alpha(\ell) - \alpha(r))\alpha(y) \geq \mathbf{0}\end{array}\right\} \textbf{Property 3}$$

Now, assume (1) $\alpha(a) \in S$ for any letter $a \in \Sigma$, (2) $\alpha(\ell) - \alpha(r) \in T$ for all $l \to r \in \mathcal{R}$, and (3) $S^*TS^* \subseteq P$ for some sets S and T, then the first line of **Property 3** can be deduced. Furthermore, if (4) $\alpha(\ell) - \alpha(r) \in U$ for all $\ell \to r \in \mathcal{S}$, and (5) $S^*US^* \subseteq N$ for some other set U, then the second line can be deduced. The latter property is usually given for free if we choose $S \subseteq N$ and $U \subseteq N$, since N is closed under multiplication.

$$\left.\begin{array}{rl} \mathit{range}(\alpha) \subseteq S;\ S \subseteq N; \forall(\ell \to r) \in \mathcal{R}, & \alpha(l) - \alpha(r) \in T;\ S^*TS^* \subseteq P \\ \forall(\ell \to r) \in \mathcal{S}, & \alpha(l) - \alpha(r) \in N \end{array}\right\} \textbf{Property 4}$$

In the original paper, the first line of **Property 4** is presented using the definition of "core of a set of matrices", namely $\mathit{core}(\mathcal{A}) = \{d \in N \mid \mathcal{A}^* d \mathcal{A}^* \subseteq P\}$. Then, if for some $\mathcal{A}$, $\mathit{range}(\alpha) \subseteq \mathcal{A}$ and $\forall(\ell \to r) \in \mathcal{R},\ \alpha(\ell) - \alpha(r) \in \mathit{core}(\mathcal{A})$ hold, $\mathcal{R}$ is strongly normalising. If additionally $\mathcal{A} \subseteq N$ and $\forall(\ell \to r) \in \mathcal{S},\ \alpha(\ell) - \alpha(r) \in N$ hold, $\mathcal{R}$ is terminating *relative* to $\mathcal{S}$.

We now introduce concrete sets of matrices, presented in the original paper, which exhibit the desired properties stated above. Here, I is a subset of matrix-indices. Note that checking whether a matrix is in such a set is computable.

$$E_I = \{d \in N \mid \forall i \in I,\ d_{i,i} > 0\} \qquad M_I = \{d \in N \mid \forall i \in I, \exists j \in I,\ d_{i,j} > 0\}$$
$$P_I = \{d \in N \mid \exists i, j \in I,\ d_{i,j} > 0\}$$

▶ **Lemma 2** ([6, Lemma 4]). $P_I = \mathit{core}(E_I), \quad M_I = \mathit{core}(M_I)$

▶ **Corollary 3.** *Let $\mathcal{R}$ and $\mathcal{S}$ be SRSs over signature Σ and let $\alpha : \Sigma \to \mathbb{Z}^{n \times n}$ be a matrix interpretation and $\emptyset \subset I \subseteq \{1, \ldots, n\}$. Let $\mathcal{S}$ satisfy $\alpha(\ell) - \alpha(r) \in N$ for all $\ell \to r \in \mathcal{S}$. Then relative termination $SN(\mathcal{R}/\mathcal{S})$ is ensured if one of the following conditions is satisfied.*

- *$\mathit{range}(\alpha) \subseteq E_I$ and $\alpha(\ell) - \alpha(r) \in P_I$ for all $\ell \to r \in \mathcal{R}$, or*
- *$\mathit{range}(\alpha) \subseteq M_I$ and $\alpha(\ell) - \alpha(r) \in M_I$ for all $\ell \to r \in \mathcal{R}$.*

Note that only the inclusions from left to right in Lemma 2 (e.g. $P_I \subseteq \mathit{core}(E_I)$) are required to get Corollary 3. Nevertheless, the original proofs of the inclusions from right to left were found to be flawed during our Isabelle formalisation. We present an instance contradicting the original proof for each inclusion before providing a reconstructed proof.

1. Original proof of $core(E_I) \subseteq P_I$ (directly quoted from [6]): For showing the inverse inclusion $P/E_I \subseteq P_I$, assume the existence of a matrix d with $dE_I \subseteq P$ and $d \notin P_I$, so $d_{i,j} = 0$ for $i, j \in I$. Define $e \in E_I$ by $e_{i,j} = 1$ for $i = j \in I$ and $e_{i,j} = 0$ otherwise. Then $de = \mathbf{0} \notin P$, a contradiction.

   Instance: Consider the set of indices $I := \{1\}$ and the matrix $d := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then, $d \notin P_I$ holds and finally, observe that $de = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in P$.
2. Similarly for $core(M_I) \subseteq M_I$: $I := \{1\}$ and $d := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
3. Reconstructed proofs: For both inclusions, the only change needed is that, instead of proving $de = \mathbf{0}$ (resp. $md = \mathbf{0}$) which is not true, we prove that $ede = \mathbf{0}$ (resp. $mdm = \mathbf{0}$) which is a contradiction with respect to the definition of *core*.

3.1 Arbitrary ordered rings

We now consider a generalisation of using integer matrices to allow matrices over other ordered rings with a strongly normalising order $\succ$ on the set of non-negative numbers. In particular we consider δ-orders [7, 8]. Here, some parameter $\delta > 0$ is fixed, and $\succ_\delta$ is defined as $x \succ_\delta y$ iff $x - y \geq \delta$. By choosing $\delta = 1$ and the integers, we obtain the original setting of [6], but we may also choose a different value of δ and the ring of rational or real numbers.

The definitions of E_I, P_I and M_I are adjusted as follows to the new setting with δ-orders, where we also add the new set $L_{I,\delta}$: for the integers with $\delta = 1$ we have $M_I = L_{I,\delta}$, so in the original setting no differentiation between these sets is required, but in the general case we gain from a distinction between these sets.

$$E_I = \{d \in N \mid \forall i \in I,\ d_{i,i} \geq 1\} \qquad M_I = \{d \in N \mid \forall i \in I, \exists j \in I,\ d_{i,j} \geq 1\}$$
$$P_{I,\delta} = \{d \in N \mid \exists i, j \in I,\ d_{i,j} \succ_\delta 0\} \qquad L_{I,\delta} = \{d \in N \mid \forall i \in I, \exists j \in I,\ d_{i,j} \succ_\delta 0\}$$

The results of [6] generalise as follows: we define P_δ as the set of non-negative matrices d for which at least one entry $d_{i,j}$ satisfies $d_{i,j} \succ_\delta 0$; the definition of the core is generalised to $core_\delta(\mathcal{A}) = \{d \in N \mid \mathcal{A}^* d \mathcal{A}^* \subseteq P_\delta\}$.

We obtain the important direction of Lemma 2 in a more general setting.

▶ **Lemma 4.** *$P_{I,\delta} \subseteq core_\delta(E_I)$, $\quad L_{I,\delta} \subseteq core_\delta(M_I)$.*

The proof of the lemma is similar to the integer setting. Here, we only mention why it was required to change the definitions of E_I and M_I. Multiplication with some number $d_{i,j}$ is monotone for δ-orders: if $x \succ_\delta y$ and $d_{i,j} \geq 1$ then also $d_{i,j}x \succ_\delta d_{i,j}y$. But only requiring $d_{i,j} \succ_\delta 0$ instead of $d_{i,j} \geq 1$ at this point is not sufficient anymore to guarantee $d_{i,j}x \succ_\delta d_{i,j}y$.

▶ **Corollary 5.** *Let $\mathcal{R}$ and $\mathcal{S}$ be SRSs over signature Σ and let $\alpha : \Sigma \to D^{n \times n}$ be a matrix interpretation over domain $D \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $\emptyset \subset I \subseteq \{1, \ldots, n\}$. Let $\delta \in D$ satisfy $0 < \delta$. Let $\mathcal{S}$ satisfy $\alpha(\ell) - \alpha(r) \in N$ for all $\ell \to r \in \mathcal{S}$. Then $SN(\mathcal{R}/\mathcal{S})$ is ensured if*

- *$range(\alpha) \subseteq E_I$ and $\alpha(\ell) - \alpha(r) \in P_{I,\delta}$ for all $\ell \to r \in \mathcal{R}$, or*
- *$range(\alpha) \subseteq M_I$ and $\alpha(\ell) - \alpha(r) \in L_{I,\delta}$ for all $\ell \to r \in \mathcal{R}$.*

3.2 Term rewrite setting

The main difference between matrix interpretations for SRSs and TRSs is that in the SRS version only matrix multiplication is used, whereas for TRSs we see both addition and multiplication. Given some n-ary function symbol f, the interpretation of f is of the form

$$\alpha(f)(x_1, \ldots, x_k) = f_0 + f_1 \cdot x_1 + \ldots + f_k \cdot x_k$$

where $f_1, \ldots, f_k$ are matrices, and f_0 is a vector (as in [4, 8]) or a matrix (as in [3]). We consider the "f_0 is a matrix" setting as it subsumes the vector setting: one can always enlarge the vector to a matrix by filling the remaining columns with zero entries. Hence, in this setting a matrix interpretation is a linear polynomial interpretation with matrices as domain.

Consequently, the interpretation of a term t leads to a linear polynomial $\alpha(t)$ with matrix coefficients, and orienting a rewrite step is similar to **Property 2**, where left-contexts are replaced by term contexts C and right-contexts are replaced by substitutions σ:

$$\forall C, \sigma,\ (\ell \to r) \in \mathcal{R} \implies \alpha(C[\ell\sigma]) - \alpha(C[r\sigma]) > \mathbf{0}$$

Since contexts C are just terms with a special variable $\Box$, the hole, we can rewrite $\alpha(C[\ell\sigma]) - \alpha(C[r\sigma])$ to $\alpha(C)\{\Box/\alpha(\ell\sigma) - \alpha(r\sigma)\}$, and by a substitution lemma the previous condition can be rewritten to a variant of **Property 3**.

$$\forall C, \vec{x},\ (\ell \to r) \in \mathcal{R} \implies \alpha(C)\{\Box/\alpha(\ell) - \alpha(r)\} > \mathbf{0}$$

where $\alpha(C)\{\Box/\alpha(\ell) - \alpha(r)\}$ is a linear polynomial over variables $\vec{x}$ that range over the domain of the matrix interpretation.

We now again want to use $core_\delta(\mathcal{A}) = \{d \in N \mid \mathcal{A}^* d \mathcal{A}^* \subseteq P_\delta\}$. In order to get rid of the context C in comparisons, the interpretation needs be chosen such that being in the core propagates via $\alpha(C)$ for all possible contexts C. This is done by demanding $f_1 \in \mathcal{A}, \ldots, f_k \in \mathcal{A}$ for each k-ary symbol f. Moreover, we identify the domain of the matrix interpretation with $\mathcal{A}$. Hence, in order to satisfy these conditions we further require the following two conditions: $f_0 \in N$ for every k-ary symbol f, and whenever f is a constant, then $f_0 \in \mathcal{A}$.

By enforcing these conditions a strict decrease of a rewrite step is now ensured by the orientation condition $\alpha(\ell) - \alpha(r) \in core_\delta(\mathcal{A})$. Note that $\alpha(\ell) - \alpha(r)$ is a linear polynomial $c_0 + c_1 x_1 + \ldots c_m x_m$ with matrix coefficients $c_0, \ldots, c_m$ over variables $x_1, \ldots, x_m$ that occur in ℓ and that range over $\mathcal{A}$. So formally, for some set of matrices M we write $\alpha(\ell) - \alpha(r) \in M$ as an abbreviation for $\forall x_1, \ldots, x_m \in \mathcal{A}.\ \alpha(\ell) - \alpha(r) \in M$, and we call $c_0, \ldots, c_m$ the coefficients of $\alpha(\ell) - \alpha(r)$. We next instantiate the abstract setting with $\mathcal{A}$ and $core_\delta(\mathcal{A})$ by E_I and $P_{I,\delta}$ (resp. M_I and $L_{I,\delta}$), i.e., the sets of matrices that are known from Section 3.1.

▶ **Theorem 6** (Core matrix interpretations for TRSs). *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs over signature $\mathcal{F}$ and let i be a matrix interpretation over domain $D \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ of dimension n and let $\emptyset \subset I \subseteq \{1, \ldots, n\}$. Let every k-ary symbol f be interpreted as $\alpha(f)(x_1, \ldots, x_k) = f_0 + f_1 \cdot x_1 + \ldots + f_k \cdot x_k$. Let $\delta \in D$ satisfy $0 < \delta$. Relative termination $SN(\mathcal{R}/\mathcal{S})$ is ensured if all of the following conditions are satisfied.*

1. *$f_i \in E_I$ (resp. M_I) for all $1 \leq i \leq k$ and k-ary symbols $f \in \mathcal{F}$*
2. *$\alpha(f)(x_1, \ldots, x_k) \in E_I$ (resp. M_i) whenever all $x_1, \ldots, x_k \in E_I$ (resp. M_I); this condition is ensured by demanding—in combination with condition 1—that*
   - **a.** *$f_0 \in N$ for each $f \in \mathcal{F}$, and*
   - **b.** *$f_0 \in E_I$ (resp. M_I) for each constant $f \in \mathcal{F}$*
3. *$\alpha(\ell) - \alpha(r) \in N$ for $\ell \to r \in \mathcal{R} \cup \mathcal{S}$; this condition is ensured by demanding that each coefficient of $\alpha(\ell) - \alpha(r)$ is in N*

4. $\alpha(\ell) - \alpha(r) \in P_{I,\delta}$ *(resp.* $L_{I,\delta}$*) for each* $\ell \to r \in \mathcal{R}$*; this condition is ensured by demanding—in combination with condition 3—that some coefficient of* $\alpha(\ell) - \alpha(r)$ *is in* $P_{I,\delta}$ *(resp.* $L_{I,\delta}$*).*

Theorem 6 is the main result of this paper and it strictly generalises the SRS setting: if every unary symbol f is interpreted by $\alpha(f)(x_1) = f_1 \cdot x_1$ then Theorem 6 is equivalent to Corollary 5, and choosing $D = \mathbb{Z}$ and $\delta = 1$ we arrive at Corollary 3.

▶ Remark 7. If one does not want to prove relative termination directly, but instead requires a reduction pair—e.g., when using dependency pairs—then condition 1 can be weakened to $f_i \in N$.

The conditions in Theorem 6 are incomparable to the monotone matrix interpretations of [4, 3, 8]. Our version has the advantage in condition 4: we allow a strict decrease for any coefficient of the linear polynomial $\alpha(\ell) - \alpha(r)$ whereas the strict decrease must happen in the constant part of $\alpha(\ell) - \alpha(r)$ in [4, 3, 8]. On the other hand, condition 2b is a new requirement in Theorem 6 which is not present in [4, 3, 8]. Roughly speaking, we require that constants must be interpreted by positive matrices, in particular they cannot be interpreted by the zero matrix as in [4, 3, 8]. Note that condition 2b cannot be dropped from Theorem 6. Consider $\mathcal{R} = \{f(x) \to g(x)\}$ and $\mathcal{S} = \{g(a) \to f(a)\}$. Clearly, $SN(\mathcal{R}/\mathcal{S})$ does not hold, but one can find a matrix interpretation that satisfies all conditions except for condition 2b, namely: $D = \mathbb{Z}$, $n = 1$, $I = \{1\}$, $\delta = 1$, $\alpha(f)(x) = 2x$, $\alpha(g)(x) = x$, and $\alpha(a) = 0$.

In the remainder of this paper, we show how Theorem 6 can be strengthened even further, i.e., by using more relaxed sufficient criteria to ensure conditions 1–4. It will turn out, that with these improved conditions, Theorem 6 even subsumes the monotone matrix interpretations of [4, 8].

First of all, we can improve the criterion for condition 4 in the M_I setting:

▶ **Lemma 8.** *Condition 4 in Theorem 6 is satisfied in the* M_I *setting, if condition 3 is satisfied and* $\forall i \in I.\ \exists c \in C.\ \exists j \in I.\ c_{ij} \succ_\delta 0$*, where* C *is the set of coefficients of* $\alpha(\ell) - \alpha(r)$*.*

This new condition is more powerful than the one of Theorem 6, since there the sufficient criterion to ensure condition 4 is equivalent to $\exists c \in C.\ \forall i \in I.\ \exists j \in I.\ c_{ij} \succ_\delta 0$ where one cannot choose different coefficients c for each row i.

For the E_I setting, we further improve the sufficient criteria for conditions 2–4. Here we use a transformation similar to [2, Section 3.4] that is known for polynomial interpretations over integers. In our setting, we basically switch from the carrier E_I to N. To this end, we define 1_I to be the matrix which is always 0, except that $(1_I)_{i,i} = 1$ whenever $i \in I$. It is easy to see that $x \in E_I$ iff $x = 1_I + y$ for some $y \in N$. We now just substitute each variable x that ranges over E_I by a variable y that ranges over N.

▶ **Lemma 9.** 2. $\forall x_1, \ldots, x_k \in E_I.\ \alpha(f)(x_1, \ldots, x_k) \in E_I$ *iff* $\forall y_1, \ldots, y_k \in N.\ \alpha(f)(1_I + y_1, \ldots, 1_I + y_k) \in E_I$*.*

3. $\alpha(\ell) - \alpha(r) \in N$ *(with variables* $x_1, \ldots, x_m$ *ranging over* E_i*) iff* $(\alpha(\ell) - \alpha(r))\{x_1/1_I + y_1, \ldots, x_m/1_I + y_m\} \in N$ *(with variables* $y_1, \ldots, y_m$ *ranging over* N*.)*
4. $\alpha(\ell) - \alpha(r) \in P_{I,\delta}$ *(with variables* $x_1, \ldots, x_m$ *ranging over* E_i*) iff* $(\alpha(\ell) - \alpha(r))\{x_1/1_I + y_1, \ldots, x_m/1_I + y_m\} \in P_{I,\delta}$ *(with variables* $y_1, \ldots, y_m$ *ranging over* N*.)*

The advantage of the switch to N via Lemma 9 is that the criteria are equivalences, and the new universally quantified conditions over N can easily be decided as follows.

▶ **Lemma 10.** *Let* p *be a linear polynomial over variables* $y_1, \ldots, y_k$*.*

- *$\forall y_1, \dots, y_k \in N.\ p(y_1, \dots, y_k) \in N$ iff all coefficients of p are in N.*
- *$\forall y_1, \dots, y_k \in N.\ p(y_1, \dots, y_k) \in E_I$ iff $\forall y_1, \dots, y_k \in N.\ p(y_1, \dots, y_k) - 1_I \in N$.*
- *$\forall y_1, \dots, y_k \in N.\ p(y_1, \dots, y_k) \in P_{I,\delta}$ iff all coefficients of p are in N and the constant part of p is in $P_{I,\delta}$.*

▶ **Example 11.** The E_I interpretation $\alpha(f)(x) = 2x - 1$ for $n = 1$ and $I = \{1\}$ is not accepted by the sufficient criterion for condition 2 in Theorem 6, but it is accepted via Lemmas 9 and 10, since $\alpha(f)(1 + y) = 2(1 + y) - 1 = 2y + 1$ is clearly in E_I when y ranges over N.

With the help of Lemmas 9 and 10 we are able to show that Theorem 6 fully subsumes matrix interpretations as they are defined by Endrullis et al. [4] and Neurauter et al. [8].

▶ **Theorem 12.** *If there is some relative termination proof via a matrix interpretation following [4, 8], then there also is a relative termination proof using Theorem 6.*

Proof. Given a strictly monotone interpretation in the style of [4, 8] with $\alpha'(f)(y_1, \dots, y_k) = f_0 + f_1 y_1 + \dots + f_k y_k$—where f_0 is interpreted as matrix—we convert it into the E_I interpretation α with $I = \{1\}$ and $\alpha(f)(x_1, \dots, x_k) := \alpha'(f)(x_1 - 1_I, \dots, x_k - 1_I) + 1_I$. We get the relationship $\alpha(t) = \alpha'(t) + 1_I$ for all ground terms t, and both interpretations define the same ordering $((\alpha(\ell) - \alpha(r))\{z_1/1_I + z_1, \dots, z_m/1_I + z_m\}) = \alpha'(\ell) - \alpha'(r))$. Moreover, all conditions of Theorem 6 are satisfied for α by using the criteria in Lemmas 9 and 10 whenever α' satisfies the criteria for a relative termination proof in [4, 8]. ◀

We briefly show that Lemma 9 is crucial for this subsumption result: given $\alpha'(f)(y) = 2y$, it is transformed into $\alpha(f)(x) = \alpha'(f)(x - 1) + 1 = 2(x - 1) + 1 = 2x - 1$, and accepting this interpretation α by Theorem 6 requires Lemma 9, cf. Example 11.

Note that the very same transformation can also be applied in the other direction, so core matrix interpretations can be turned into the ones of [4, 8] for the $E_{\{1\}}$ setting.

We do not know, whether similar transformations are possible for the M_I setting. Here, the complication arises that we cannot add or subtract a matrix to switch between M_I and N, since there is not a unique minimal element such as 1_I in the E_I case, but there are $|I|^{|I|}$ many of those: a matrix A is minimal in M_I if for every $i \in I$ there is exactly one $j \in I$ such that $A_{ij} = 1$, and all other entries are 0.

We leave it as future work, to include further sets than E_I and M_I into the formalization and into CeTA. Such an addition can be triggered if there is demand from the tool author side.

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