A New Proof for Soundness of Right-Forward Closures for Termination Analysis

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Abstract

We consider the termination problem of term rewrite systems (TRSs). Dershowitz proved that termination starting from arbitrary terms is equivalent to termination starting from all terms in the right-forward closure (RFC), provided that the TRS is right-linear or orthogonal. In this paper we provide a new proof of this result, and also include a later result that one can weaken orthogonality to locally confluent overlay TRSs. All proofs have been verified in Isabelle/HOL.

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1 Introduction

A relation $\to$ is strongly normalizing w.r.t. a set of starting objects T, written $SN(\to, T)$, iff there is no infinite sequence

$$t_0 \to t_1 \to t_2 \to t_3 \to \dots \qquad (\star)$$

with $t_0 \in T$. If there is no restriction on the set of starting objects we just write $SN(\to)$.

In this paper we consider termination of first-order unsorted term rewrite systems $\mathcal{R}$ where $\to$ is the rewrite relation $\to_\mathcal{R}$ of $\mathcal{R}$ and T is either the set of all terms, or $RFC(\mathcal{R})$, the right-hand sides of forward closure of $\mathcal{R}$, also known as the right-forward closure of $\mathcal{R}$.

This paper will illustrate a novel proof of the following result of Dershowitz, and this theorem will also be generalized to a larger class of TRSs.

▶ **Theorem 1** ([3]). *Let $\mathcal{R}$ be a right-linear or orthogonal TRS. Then*

$$SN(\to_\mathcal{R}) \longleftrightarrow SN(\to_\mathcal{R}, RFC(\mathcal{R})).$$

Note that Theorem 1 has applications in termination proving, e.g., in combination with the match-bounds technique [5, 6, 10]. Here, the change from all starting terms to just $RFC(\mathcal{R})$ can become crucial for a successful termination proof.

There is a challenge in verifying the proof of Theorem 1, e.g., in a formal setting.

One of the problems is that the definition of $RFC(\mathcal{R})$ as it is understood nowadays (defined via narrowing), differs from Dershowitz' original definition [3] (defined via infinite chains, where in infinite derivations—such as ($\star$)—active positions are marked and have to satisfy certain conditions). Moreover, the original proof stays at an intuitive level, e.g., by claiming that certain steps can be reordered in an infinite derivation, without providing further details.

Dershowitz and Hoot provide an alternative definition of forward closures [4, Definition 6] that is based on narrowing, and there is a clear correspondence to $RFC(\mathcal{R})$. Unfortunately, the connection between infinite chains [3] and *infinite* forward closures as they occur in [4, Theorem 6] remains at an intuitive level in this paper. Note that this paper also includes an improved result: Theorem 1 is still true if one replaces *orthogonal TRS* by *locally confluent overlay TRS* [4, Theorem 6].

The definition of *infinite* forward closures is made formal by Geupel. He also provides a detailed proof of Theorem 1 for non-overlapping TRSs [8, Theorem 2], but does not consider right-linear TRSs.

A detailed proof of Theorem 1 for the case of right-linear TRSs is partly provided by Zantema [13, Section 6]: he restricts the theorem to string rewrite systems.

Our main contributions are:

- The development of a novel proof of Theorem 1 for a definition of $\mathit{RFC}(\mathcal{R})$ that is based on narrowing, with the inclusion of locally confluent overlay TRSs.
- The proof does not require any reordering of steps as in the original proof.
- The full proof is formally verified in Isabelle/HOL, cf. IsaFoR version 3.5 [12].

2 Preliminaries

We assume familiarity with term rewriting [2] and recall important notions and notations.

We consider first order terms $s, t, \ell, r, \ldots \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ that consist of variables $x, y, z, \ldots \in \mathcal{V}$ and function applications $f(t_1, \ldots, t_n)$ for n-ary symbols $f \in \mathcal{F}$. Substitutions $\sigma, \delta, \mu, \ldots$ are mappings from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and we write $t\sigma$ to substitute each variable x in a term t by $\sigma(x)$. Given a substitution σ and some set of variables X we write $\sigma \upharpoonright X$ for the substitution that restricts σ to X, i.e., $(\sigma \upharpoonright X)(x) = \sigma(x)$ if $x \in X$, and $(\sigma \upharpoonright X)(x) = x$ if $x \notin X$. We denote the set of positions of t by $\mathit{Pos}(t)$, $t|_p$ is the subterm of t at position p and $t[s]_p$ replaces $t|_p$ by s in t at position p. The set $\mathit{FPos}(t)$ is the set of function positions, i.e., $\mathit{FPos}(t) = \{p \in \mathit{Pos}(t) \mid t|_p \notin \mathcal{V}\}$. The strict subterm relation is denoted by $\rhd$, i.e., $s \rhd t$ iff $s \neq t$ and $s|_p = t$ for some $p \in \mathit{Pos}(s)$. Term t is linear, if no variable occurs more than once in t; $\mathit{Vars}(t)$ is the set of variables of term t. We write $\mathit{mgu}(s, t) = \sigma$ if σ is a most general unifier of s and t.

A TRS $\mathcal{R}$ is a set of rules $\ell \to r$ such that $\ell \notin \mathcal{V}$ and $\mathit{Vars}(\ell) \supseteq \mathit{Vars}(r)$. The rewrite relation of $\mathcal{R}$ is defined as $s \to_{\mathcal{R},p} t$ iff $p \in \mathit{FPos}(t)$, $\ell \to r \in \mathcal{R}$, $s|_p = \ell\mu$, and $t = s[r\mu]_p$ for some p, ℓ, r, μ. Narrowing of $\mathcal{R}$ is defined as $s \rightsquigarrow_{\mathcal{R},\mu \upharpoonright \mathit{Vars}(s),p} t$ iff $p \in \mathit{FPos}(t)$, $\ell \to r \in \mathcal{R}$, $\mathit{mgu}(s|_p, \ell) = \mu$, and $t = s[r]_p\mu$, for some p, ℓ, r, μ where—in contrast to the rewrite relation—unification is used instead of matching. For narrowing it is always assumed that the variables of rule $\ell \to r$ are renamed apart so that they are disjoint to $\mathit{Vars}(s)$. We often omit the subscripts in the rewrite and narrowing relation if these are not relevant or if they are clear from the context.

The set of right-hand sides of a TRS $\mathcal{R}$ is denoted by $\mathit{rhs}(\mathcal{R})$, and $\mathcal{R}$ is right-linear if all terms in $\mathit{rhs}(\mathcal{R})$ are linear. We refer to the definition of critical pairs to the textbook [2]. An *orthogonal* TRS does not have any critical pairs. In an *overlay* TRS all critical pairs stem from overlaps at the root. A TRS is *locally confluent* if all its critical pairs are joinable.

We are now able to define the right-forward closure of a TRS and recast Dershowitz' and our result using the notations that have been introduced.

▶ **Definition 2** ($\mathit{RFC}(\mathcal{R})$). *$\mathit{RFC}(\mathcal{R})$ is the least set that contains $\mathit{rhs}(\mathcal{R})$ and is closed under narrowing.*

$$\mathit{RFC}(\mathcal{R}) = \{t \mid s \in \mathit{rhs}(R) \wedge s \rightsquigarrow_{\mathcal{R}}^* t\} = \rightsquigarrow_{\mathcal{R}}^*(\mathit{rhs}(\mathcal{R}))$$

▶ **Theorem 3** ([3]). *Let $\mathcal{R}$ be a right-linear or orthogonal TRS. Then $\mathit{SN}(\to_{\mathcal{R}})$ is satisfied, if $\mathit{SN}(\to_{\mathcal{R}}, \mathit{RFC}(\mathcal{R}))$ or $\mathit{SN}(\rightsquigarrow_{\mathcal{R}}, \mathit{rhs}(\mathcal{R}))$.*

▶ **Theorem 4** ([4, 12]). *Theorem 3 is still valid, if one weakens "orthogonal TRS" to "locally confluent overlay TRS".*

▶ **Example 5.** A subset of Toyama's TRS is easily proved to be terminating by Theorems 3 and 4. For the TRS $\mathcal{R} = \{f(a,b,x) \to f(x,x,x), g(x,y) \to x\}$ we obtain $\mathit{rhs}(\mathcal{R}) = \{f(x,x,x), x\}$. Since both terms in $\mathit{rhs}(\mathcal{R})$ are normal forms w.r.t. narrowing, in particular $\mathit{SN}(\rightsquigarrow_{\mathcal{R}}, \mathit{rhs}(\mathcal{R}))$ is satisfied. Thus, by Theorem 3 (or by the stronger Theorem 4) we may conclude termination of $\mathcal{R}$.

▶ **Example 6.** The full version of Toyama's TRS $\{f(a,b,x) \to f(x,x,x), g(x,y) \to x, g(x,y) \to y\}$ is non-terminating. This TRS shows that one cannot just drop the pre-conditions on the TRS in Theorems 3 and 4. The reason is that—similarly to Example 5—narrowing is terminating if one starts from an arbitrary right-hand side of this TRS.

3 A Novel Proof of Theorem 3 and of Theorem 4

Before we start with the main proof of Theorem 4, let us first state some easy connections between rewriting and narrowing.

▶ **Lemma 7.**
- *If $s \to_p t$ then $s \rightsquigarrow_{\mu,p} t\mu$ for some variable renaming μ.*
- *If $s \rightsquigarrow_{\mu,p} t$ then $s\mu \to_p t$.*
- *If $s \rightsquigarrow_{\mu,p} t$ for some variable renaming μ then $s \to_p t\mu^{-1}$.*

The first property of Lemma 7 shows that narrowing can simulate rewriting; the second property is for the reverse direction, provided that one instantiates the starting term s by the substitution μ from the narrowing step; and finally, if narrowing just uses variable renamings μ, then it is basically rewriting (modulo variable renamings).

Note that $\mathit{SN}(\to_{\mathcal{R}})$ is equivalent to $\mathit{SN}(\to_{\mathcal{R}}, \sigma(\mathit{rhs}(\mathcal{R})))$, i.e., for termination analysis it suffices to consider all instances $r\sigma$ of right-hand sides r of $\mathcal{R}$: this can be seen by using a minimal non-terminating term argument as it is used for dependency pairs [1].

The idea of the RFC termination argument is now that narrowing can recover σ, so one just needs to start from some right-hand side and will then piecewise reconstruct σ to simulate the non-terminating sequence, and at some point all narrowing steps will become rewrite steps as in the third property of Lemma 7. To this end, we need the following result which permits us to connect a rewrite step with source $s\sigma$ with a narrowing step with source s, provided that the position of the rewrite step is within s: one can decompose σ into $\mu\sigma'$ and the rewrite step is already possible with $s\mu$.

▶ **Lemma 8** (One-step simulation). *If $s\sigma \to_p t$ and $p \in \mathit{FPos}(s)$ then there are s' and μ and σ' such that*

$$s \rightsquigarrow_{\mu,p} s' \text{ and } \sigma = \mu\sigma' \text{ and } t = s'\sigma'$$

or illustrated a bit differently where now the rewrite step is underlined:

$$s\sigma = \underline{s\mu}\sigma' \to \underline{s'}\sigma' = t \quad \text{and} \quad s \rightsquigarrow_{\mu} s'$$

Now assume there is some non-terminating derivation

$$s_0\sigma_0 = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \dots \tag{1}$$

The aim is to find some invariant on s and σ such that it is satisfied for s_0 and σ_0. Moreover, the invariant must be preserved by Lemma 8, i.e., the terms s' and substitution σ' of this

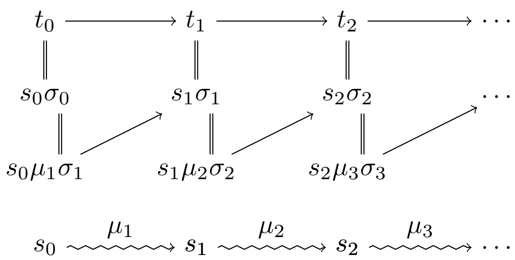

Figure 1 Infinite Application of One-Step Simulation

lemma should also satisfy it. The reason is that with such an invariant an infinite narrowing sequence can be constructed as in Figure 1. Moreover, one can further prove that eventually all narrowing steps must be rewrite steps. Thus, $\neg SN(\rightsquigarrow_{\mathcal{R}}, \{s_0\})$ and $\neg SN(\rightarrow_{\mathcal{R}}, \rightsquigarrow_{\mathcal{R}} (\{s_0\}))$ can be derived, and hence both $SN(\rightsquigarrow_{\mathcal{R}}, \{s_0\})$ and $SN(\rightarrow_{\mathcal{R}}, \rightsquigarrow_{\mathcal{R}}^* (\{s_0\}))$ are sufficient criteria for proving $SN(\rightarrow_{\mathcal{R}}, \{s_0\sigma_0\})$.

▶ **Lemma 9.** *Let I be some invariant such that all of the following is satisfied:*

- *there is an infinite derivation as in (1),*
- *$I(s_0, \sigma_0)$,*
- *if $I(s, \sigma)$ and $s\sigma \rightarrow_p t$ then $p \in FPos(s)$, and*
- *if $I(s, \sigma)$ and $s \rightsquigarrow_\mu s'$ and $\sigma = \mu\sigma'$ then $I(s', \sigma')$.*

Then $\neg SN(\rightsquigarrow_{\mathcal{R}}, \{s_0\})$ and $\neg SN(\rightarrow_{\mathcal{R}}, \rightsquigarrow_{\mathcal{R}}^* (\{s_0\}))$.

Proof. The proof works as follows. First, by using the invariant one can indeed construct the infinite narrowing sequence as indicated in Figure 1. This already proves $\neg SN(\rightsquigarrow_{\mathcal{R}}, \{s_0\})$. Afterwards, it only needs to be shown that eventually all μ_i are variable renamings, so that the narrowing steps in Figure 1 can eventually be turned into rewrite steps by Lemma 7.

To this end, observe that

$$t_0 = s_0\sigma_0 = s_0\mu_1\sigma_1 = s_0\mu_1\mu_2\sigma_2 = s_0\mu_1\mu_2\mu_3\sigma_3 = \dots.$$

Thus, $s_0\mu_1\mu_2 \dots \mu_n$ is an instance of t_0 for all n. Therefore, eventually all μ_i are variable substitutions, i.e., $\mu_i : \mathcal{V} \rightarrow \mathcal{V}$ for almost every i. For these variable substitutions we infer

$$\mathit{Vars}(s_{i+1}\mu_{i+1}) = \mu_{i+1}(\mathit{Vars}(s_{i+1})) \subseteq \mu_{i+1}(\mathit{Vars}(s_i\mu_i))$$

because $s_i\mu_i \rightarrow s_{i+1}$ implies $\mathit{Vars}(s_i\mu_i) \supseteq \mathit{Vars}(s_{i+1})$, and hence

$$|\mathit{Vars}(s_{i+1}\mu_{i+1})| \leq |\mu_{i+1}(\mathit{Vars}(s_i\mu_i))| \leq |\mathit{Vars}(s_i\mu_i)|$$

shows that $|\mathit{Vars}(s_i\mu_i)|$ weakly decreases with increasing i. Thus, at some point $|\mathit{Vars}(s_i\mu_i)|$ becomes constant, and hence for all future i, μ_{i+1} is a variable renaming. ◀

We now only need to find suitable invariants I to obtain Theorem 4.

Proof of Theorem 4: locally confluent overlay TRSs. Proving this second part of Theorem 4 is quite simple with the help of Lemma 9: we define I such that $I(s, \sigma)$ is satisfied if σ is a normal form substitution, i.e., $\sigma(x)$ is in normal form w.r.t. $\rightarrow_{\mathcal{R}}$ for all x. Hence, by

Lemma 9 we conclude that both $SN(\rightsquigarrow_{\mathcal{R}}, \{s\})$ and $SN(\rightarrow_{\mathcal{R}}, \rightsquigarrow^*_{\mathcal{R}}(\{s\}))$ imply $SN(\rightarrow_{\mathcal{R}}, \{s\sigma\})$ for all normal form substitutions σ, or replacing $\{s\}$ by $rhs(\mathcal{R})$ we arrive at:

$$SN(\rightsquigarrow_{\mathcal{R}}, rhs(\mathcal{R})) \vee SN(\rightarrow_{\mathcal{R}}, RFC(\mathcal{R})) \quad \text{implies} \quad SN(\rightarrow_{\mathcal{R}}, \sigma(rhs(\mathcal{R})))$$

The switch from $SN(\rightarrow_{\mathcal{R}}, \sigma(rhs(\mathcal{R})))$ for all normal form substitutions σ to full $SN(\rightarrow_{\mathcal{R}})$ is now performed using a result of Gramlich [9]: for locally confluent overlay TRSs, innermost termination and termination coincide, and thus the restriction to normal form substitution can be assumed without loss of generality. ◀

Proof of Theorem 4: right-linear TRSs. For this part of the proof we define I differently, namely $I(s, \sigma)$ is satisfied if s is linear and σ is a strongly normalizing substitution, i.e., $SN(\rightarrow_{\mathcal{R}}, \{\sigma(x) \mid x \in \mathcal{V}\})$.

Given some infinite derivation (1), we cannot immediately apply Lemma 9, but first have to do some preprocessing as follows.

For an arbitrary rewrite step $s\sigma \rightarrow_p t$, it is either the case that $p \in FPos(s)$ (and we use Lemma 8), or the rewrite step is completely inside σ, e.g., below variable x. If s is linear, then in the latter case the step $s\sigma \rightarrow_p t$ can be simulated by narrowing with zero steps as follows: $s \rightsquigarrow^0 s$ and $\sigma \rightarrow^+ \sigma'$ where σ' is defined as the substitution that is obtained by rewriting $x\sigma$ to the corresponding subterm of t. In this way one obtains $s\sigma \rightarrow t = s\sigma'$. So in both cases ($p \in FPos(s)$ is satisfied or not), one can find a term s' and substitution σ' such that $t = s'\sigma'$ and $I(s', \sigma')$ is satisfied. Moreover, $s \rightsquigarrow^* s'$. Hence, we can again obtain an infinite simulation of the rewrite sequence by narrowing steps as indicated in Figure 1, except that now some of the narrowing steps need to be replaced by equalities.

Note that we further obtain $\sigma_i(Vars(s_i))\ (\rightarrow_{\mathcal{R}} \cup \rhd)^*_{mul}\ \sigma_{i+1}(Vars(s_{i+1}))$ where here $Vars(s_i)$ is interpreted as the multiset of variables of a term, and $(\rightarrow_{\mathcal{R}} \cup \rhd)_{mul}$ is the multiset extension of relation $\rightarrow_{\mathcal{R}} \cup \rhd$. Since I enforces that the substitutions are strongly normalizing, there must be some point k such indeed a strict decrease is not possible anymore, i.e., for all $i \geq k$, the relation $\sigma_i(Vars(s_i))\ (\rightarrow_{\mathcal{R}} \cup \rhd)^+_{mul}\ \sigma_{i+1}(Vars(s_{i+1}))$ is not satisfied. Since such a strict decrease is always obtained if the rewrite step is completely inside the substitution, we know that for each $i \geq k$, indeed the position of the rewrite step must be in s_i. Hence, we can apply Lemma 9 on the infinite derivation $t_k = s_k\sigma_k \rightarrow_{\mathcal{R}} s_{k+1}\sigma_{k+1} \rightarrow_{\mathcal{R}} \dots$ and obtain $\neg SN(\rightsquigarrow_{\mathcal{R}}, \{s_k\})$ and $\neg SN(\rightarrow_{\mathcal{R}}, \rightsquigarrow^*_{\mathcal{R}}(\{s_k\}))$. In combination with $s_0 \rightsquigarrow^* s_k$ this nearly completes the proof as in the case for locally confluent overlay TRSs. There are two additional steps that still need to proven.

First, we argue that invariant I is initially satisfied. To this end, we again refer to the minimal non-terminating term argument: if $\mathcal{R}$ is not terminating, then there must be a minimal non-terminating term u w.r.t. the subterm relation. Any infinite derivation must make a root step at some point, so $u \rightarrow^*_{\mathcal{R}} \ell\sigma \rightarrow_{\mathcal{R}} r\sigma$ for some rule $\ell \rightarrow r \in \mathcal{R}$, and $r\sigma$ is non-terminating. By minimality of u it is easy to see that $I(r, \sigma)$ is satisfied, and we choose $s_0 = r$ and $\sigma_0 = \sigma$ as starting point of derivation (1).

The final issue is the preservation of the invariant, i.e., in particular we must show for every narrowing step $s_i \rightsquigarrow_{\mathcal{R}} s_{i+1}$ that linearity of s_i is carried over to linearity of s_{i+1}. This fact is stated in the upcoming Lemma 11. ◀

Before we show that narrowing preserves linearity, we first need the result that unification preservers linearity. Note that not all participating terms have to be linear, which is crucial to obtain linearity preservation of narrowing for TRSs that are right-linear, but not necessarily left-linear.

▶ **Lemma 10** (Unification Involving Linear Terms, [11]). *Let $\mathit{Vars}(s) \cap \mathit{Vars}(t) = \mathit{Vars}(u) \cap \mathit{Vars}(t) = \emptyset$, let t and u be linear. If s and t are unifiable with $mgu(s,t) = \sigma$ then $u\sigma$ is linear.*

Lemma 10 allows non-linearity of the left term s of the unification problem (s,t), whereas the right term t must be linear. This lemma is proven by generalizing (s,t) to multisets of such unification problems, and then following the computation of a swap-free unification algorithm. In such an algorithm there is no transition of the form that replaces (s,t) by (t,s), with the overhead that there must be two rules for treating substitutions, one for pairs (x,t) and one for (t,x). Being swap-free is important to maintain the invariant regarding linearity of the right-hand sides of a unification problem.

▶ **Lemma 11** (Linearity Preservation of Narrowing, [11]). *If $v \rightsquigarrow_{\mathcal{R},p} w$ and $\mathcal{R}$ is right-linear and v is linear, then also w is linear.*

Proof. We mainly need to choose suitable parameters in Lemma 10 in order to prove that w is linear: we instantiate s by ℓ, t by $v|_p$, u by $v[r]_p$, and σ by μ. Here $\ell \to r$ and μ are the applied rule and the unifier from the narrowing step, respectively. ◀

It remains as future work to verify computable approximations of $\mathit{RFC}(\mathcal{R})$ (one might start with string rewrite systems [13, Section 6], extend this to linear TRSs [7, Section 8] and even further to right-linear TRSs [10, Section 8]) and then combine these approximations with CeTA's checker for match-bounds in order to support RFC match-bounds in CeTA.

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