Disjunctive Termination for Affluent Families

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Abstract

We introduce *affluence*, a condition on composition that entails a finite *family* of rewrite relations is terminating iff its *union* is (disjunctive termination), and relate it to *jumping*. Our proofs transform infinite reductions in the family union into such in family members, by induction on family size.

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1 Introduction

We model program execution by means of rewrite systems, where we leave the objects abstract in order to not commit to any particular execution model. Suppose a program P comprises a finite family $(P)_I$ of program modules P_i for $i \in I$, such that the rewrite relation $\to$ modelling P is the union $\bigcup(\to)_I$ of the finite family $(\to)_I$ of rewrite relations modelling $(P)_I$. It is good science to aim for modularity, here, to try to show termination of P as a consequence of termination of (each module in) the family $(P)_I$. Naïvely taken, this fails: both $a \triangleright b$ and $b \blacktriangleright a$ are terminating, but their union $\to := \triangleright \cup \blacktriangleright$ is not, allowing the reduction cycle $a \to b \to a$. The example exhibits feature interaction: composing steps from the different modules $\triangleright$ and $\blacktriangleright$ was not accounted for, and indeed led to non-termination. Therefore, to account for that executions of P arise by composing executions of its modules in $(P)_I$, it is natural to make additional *assumptions on compositions*. We discuss two such assumptions, *affluence* and *jumping*, showing statements for them of shape: for a finite family $(\to)_I$ and $\to := \bigcup(\to)_I$, if there's a $\to$-reduction γ having property $\mathcal{P}$, then there's a $\to_i$-reduction δ having property $\mathcal{Q}$ for *some* $i \in I$. Our proofs *transform* γ into δ and are by *induction* on the family size. Letting both $\mathcal{P}$ and $\mathcal{Q}$ express that the reduction is infinite, it then follows by contraposition that $\to$ is terminating if $(\to)_I$ is, giving our main applications.

We use arrow-like notations $\to, \blacktriangleright, \triangleright, \dots$ to denote *rewrite* relations, binary endorelations, and $\gamma, \delta, \epsilon, \dots$ to range over *reductions*, sequences of consecutive steps of such, denoted by repeated-arrows like $\twoheadrightarrow, \blacktriangleright\!\!\blacktriangleright, \triangleright\!\!\triangleright, \dots$. See the literature, *e.g.*, [8], for more on rewriting.

2 Affluent families

▶ **Definition 1** ([5, Def. 2.3]). $\blacktriangleright, \triangleright$ *is* affluent *if* $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup \blacktriangleright$.

Enforcing affluence bars the monster in Sec. 1 since $a \not\twoheadrightarrow a$ though $a \triangleright b \blacktriangleright a$. Affluence is *rich*: (i) If $\blacktriangleright = \triangleright$ it expresses *transitivy*; (ii) confluence (*flowing together*) $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$ is strengthened by affluence (*flowing toward*, a river being a tributary), $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadleftarrow \cup \twoheadrightarrow$, when instantiating $\triangleright$ and $\blacktriangleright$ with inverse reductions $\twoheadleftarrow$ respectively reductions $\twoheadrightarrow$; (iii) For less-than-or-equal $\leq$ on the natural numbers the assumption $n \geq \cdot \leq m$ holds for *any* n, m as $0 \leq n, m$, so affluence expresses *totality*: $n \geq m$ or $n \leq m$; (iv) The prefix order $\sqsubseteq$ on finite $\to$-reductions is affluent, *i.e.* $\sqsupseteq, \sqsubseteq$ is, iff $\to$ is *deterministic* (*cf.* the CompCert formalisation). The intuition is that affluence affords to compress consecutive out-of-order steps to yield a reduction that is progressive ($\blacktriangleright$-steps occur before $\triangleright$-steps in the reduction) and preferential (given an object in the reduction, $\blacktriangleright$-steps from it are preferred over $\triangleright$-steps). To generalise that from two rewrite relations $\blacktriangleright$ and $\triangleright$ to finite families later, we relativize affluence.

▶ **Definition 2.** *Given a $\to$-reduction γ, $\restriction\gamma$ denotes restricting a rewrite relation to objects in γ and steps to be co-initial to some step in γ, and $\blacktriangleright, \triangleright$ is* affluent for *a $\to$-reduction γ if $\blacktriangleright\restriction\gamma, \triangleright\restriction\gamma$ is affluent, where $\to = \blacktriangleright \cup \triangleright$.*

Affluence entails affluence for any reduction γ: if $a\ ((\triangleright\restriction\gamma) \cdot (\blacktriangleright\restriction\gamma))\ b$, then a, b in γ and $a\ (\triangleright \cdot \blacktriangleright)\ b$ by definition of restriction so $a\ (\triangleright \cup \blacktriangleright)\ b$ by assumption, so $a\ ((\triangleright\restriction\gamma) \cup (\blacktriangleright\restriction\gamma))\ b$ by definition. Observe that if $\to\restriction\gamma = \to$ then $\to$ has at most one normal form, which if it exists is the target of γ and thus, if γ is infinite any maximal reduction[1] is infinite too.

▶ **Definition 3.** *A reduction δ is* progressive *if $\blacktriangleright$-steps precede $\triangleright$-steps in δ except possibly for an infinite $\blacktriangleright$-tail, and* preferential *if $\triangleright$-steps in δ are from $\blacktriangleright$-normal forms. For $\to\restriction\gamma = \to = \blacktriangleright \cup \triangleright$ a reduction γ* upgrades *to δ, denoted by $\gamma\nearrow\delta$ (*promotes *to δ, denoted by $\gamma\Nearrow\delta$) if δ is co-initial to γ, maximal and progressive (and preferential).*

Observe if $\gamma\nearrow\delta$ then δ has shape $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^\alpha$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^\omega$ for $\alpha \leq \omega$ with $\alpha = \omega$ if γ is infinite.

▶ **Lemma 4.** *For a reduction γ with $\to\restriction\gamma = \to = \blacktriangleright \cup \triangleright$ and $\blacktriangleright, \triangleright$ affluent, $\gamma\Nearrow\hat{\gamma}$ for some $\hat{\gamma}$.*

Proof. Under the assumptions, let δ be a maximal $\blacktriangleright$-reduction co-initial to γ. If δ is infinite, then $\gamma\Nearrow\hat{\gamma}$ for $\hat{\gamma} := \delta$. Otherwise, we construct a $\triangleright$-reduction ϵ by initially setting it to the empty reduction on the target of δ and repeating as long as its target is not that of γ and in $\blacktriangleright$-normal form, to append to ϵ some $\triangleright$-step to an object that is either a $\blacktriangleright$-normal form or non-$\blacktriangleright$-terminating, which we claim exists. ▬ If ϵ is infinite, then per construction $\gamma\Nearrow\delta \cdot \epsilon$, so we set $\hat{\gamma}$ to $\delta \cdot \epsilon$, as visualised in Fig. 1 for reductions δ and ϵ, with $\lightning$ marking $\blacktriangleright$-normal forms in γ; ▬ If ϵ is finite, its target is either non-$\blacktriangleright$-terminating or the target of γ and we have $\gamma\Nearrow\hat{\gamma}$ when setting $\hat{\gamma}$ to $\delta \cdot \epsilon$, in the former case followed by any infinite $\blacktriangleright$-reduction.

To prove the claim, note an object $\hat{b}$ being in $\blacktriangleright$-normal form and not the target of γ, has a step $\hat{b} \triangleright c$ for some c. If c is in $\blacktriangleright$-normal form we return the $\triangleright$-step to it. Otherwise, $c \blacktriangleright c'$ for some c'. By $\hat{b} \triangleright c \blacktriangleright c'$ and affluence, we have $\hat{b} \triangleright c'$ as we cannot have the other disjunct[2] by $\hat{b}$ being in $\blacktriangleright$-normal form. Repeating this for c' instead of c, we eventually end up in the first case or find an infinite reduction $c \blacktriangleright c' \blacktriangleright \ldots$ so return the $\triangleright$-step to c. ◀

Assuming a non-terminating $\to$-reduction γ were to exist, restricting affluent $\blacktriangleright, \triangleright$ to γ allows one to conclude to non-termination of $\blacktriangleright$ or $\triangleright$ by Lem. 4 (using the observations), so:

▶ **Corollary 5.** *Let $\to := \blacktriangleright \cup \triangleright$. $\blacktriangleright, \triangleright$ are terminating iff $\to$ is, if:* **1.** *$\blacktriangleright, \triangleright$ is affluent [2, 1];* **2.** *$\to \cdot \to \subseteq \to$ (transitivity) [3, pp. 31,32][2, 8, 6, 1].*

[1] A reduction is *maximal* if it is infinite or ends in a normal form [8]. *Computations* in [6] are maximal.
[2] That is, we cannot have the other *operand of the union* in the definition of affluence, Def. 1 here $\hat{b} \blacktriangleright \ldots$.

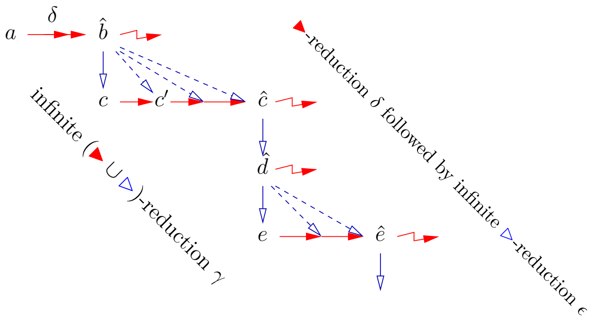

Figure 1 Transformation of γ into $\delta \cdot \epsilon$ of shape $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$ in the proof of Lem. 4 with $\gamma \Nearrow \delta \cdot \epsilon$

We extend the above to finite families $(\to)_I$ of rewrite relations with I ordered by $<$.

▶ **Definition 6.** *For $\mathcal{F} := (\to)_I$ a family of rewrite relations, $\mathcal{F}$ is* affluent *if $\to_{>i} \cdot \to_i \subseteq \to := \bigcup \mathcal{F}$ for $i \in I$, and* affluent *for a reduction γ if $(\to\restriction\gamma)_I$ is affluent.*

Affluence entails affluence for any reduction γ and $\bigcup(\to\restriction\gamma)_I = (\bigcup \mathcal{F})\restriction\gamma$, by distributivity. From now on we let index-sets range over intervals $[\ell, n]$ of natural numbers. A reduction δ is *progressive* if $\to_i$-steps precede $\to_j$-steps in δ for $i < j$, except possibly for an infinite $\to_k$-tail for some k, so has shape $\twoheadrightarrow_\ell \cdot \ldots \cdot \twoheadrightarrow_n (\cdot \to_k^{\omega})$ with the last part optional, and *preferential* if from any source of a $\to_j$-step in δ there's no $\to_i$-step, for $i < j$.

▶ **Theorem 7.** *Let γ be a reduction with $\to\restriction\gamma = \to = \bigcup \mathcal{F}$ with $\mathcal{F} := (\to)_{[\ell,n]}$ and let $\nearrow$ be defined as in Def. 3 under these assumptions. If $\mathcal{F}$ is affluent, then $\gamma \nearrow \epsilon$ for some ϵ.*

Proof. By induction on $n \dot{-} \ell$ with base case Lem. 4. Otherwise, the $\to$-reduction γ can be seen as a $(\blacktriangleright \cup \triangleright)$-reduction for $\blacktriangleright := \to_\ell$ and $\triangleright := \bigcup(\to)_J$ with $J := [\ell+1, n]$. Then $\blacktriangleright, \triangleright$ is affluent, so $\gamma \Nearrow \hat{\gamma}$ by Lem. 4 with $\hat{\gamma}$ of shape either $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ or $\blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\alpha}$ for $\alpha \leq \omega$. In either case let δ be the $\triangleright$-subreduction of $\hat{\gamma}$. The family $(\to\restriction\delta)_J$ is affluent for δ since not only are the objects in δ objects in γ by $\gamma \Nearrow \hat{\gamma}$, they are (when source of a step in δ) in $\blacktriangleright\restriction\gamma$-normal form, entailing $(\to\restriction\delta)_J$-steps compose to other such, cannot compose to $\blacktriangleright$-steps (*Cf.* the proof of the claim in Lem. 4, footnote 2). Hence the IH applies to δ yielding $\delta \nearrow \hat{\delta}$ with $\hat{\delta}$ of shape $\twoheadrightarrow_{\ell+1} \cdot \ldots \cdot \twoheadrightarrow_n (\cdot \to_k^{\omega})$ with the last infinite part (for some $\ell + 1 \leq k \leq n$) optional. Setting ϵ to the reduction obtained by substituting[3] $\hat{\delta}$ for δ in $\hat{\gamma}$ we conclude to $\gamma \nearrow \epsilon$. ◀

▶ **Corollary 8.** *Let $\mathcal{F} := (\to)_I$ and $\to := \bigcup \mathcal{F}$.* **1.** *$\mathcal{F}$ is terminating iff $\to$ is, for affluent $\mathcal{F}$;* **2.** *$\mathcal{F} = (\blacktriangleright, \triangleright, \gg)$ is terminating iff $\to$ is, if $(\triangleright \cdot \blacktriangleright) \cup (\gg \cdot \blacktriangleright) \cup (\gg \cdot \triangleright) \subseteq \to$ [1, Thm. 2];* **3.** *$\mathcal{F}$ is terminating iff $\to$ is, if $\to \cdot \to \subseteq \to$ (transitivity; disjunctive termination) [6].*

Whereas Cor. 8(3) cannot be proven directly from Cor. 5(2) **by induction** on family size, as argued on [7, p. 1218], it does not follow as implied there that one can't proceed by induction: we showed Thm. 7 from Lem. 4 **inductively**, having the former as consequences.

3 Jumping families

Sec. 2 was written such that its development (Def. 2,Def. 3,Lem. 4,Def. 6,Thm. 7) is preserved when **replacing** *affluence* and $\restriction$ everywhere by *jumping* and $|$, both defined next, where $|$ relaxes restriction $\restriction$ (Def. 2) to accommodate that jumping is *weaker*[4] than affluence.

[3] We use the convention that concatenating to an infinite reduction yields the infinite reduction.
[4] It can be further weakened to *yumping*, by adding $\blacktriangleright \cdot \to^{\omega}$ as a third disjunct to its right-hand side.

▶ **Definition 9.** *$\blacktriangleright, \triangleright$ is* jumping *if $\triangleright \cdot \blacktriangleright \subseteq \triangleright \cup (\blacktriangleright \cdot \twoheadrightarrow)$ for $\to := \blacktriangleright \cup \triangleright$ [2, 1]. For $\to$-reduction γ, $|\gamma$ restricts a rewrite relation to objects c* along *γ,* i.e. *to objects c such that the source of γ reduces to c and c reduces to the target of γ (if any) or is not terminating.*

Jumping entails jumping for any reduction γ using that if $\delta : a \twoheadrightarrow b$ for a, b along γ then all c in δ are along γ. If $\to|\gamma = \to$, only the target of γ (if any) can be a normal form, since if c is non-terminating, then *some* d with $c \to d$ is non-terminating too.

▶ **Lemma 10.** *For reduction γ with $\to|\gamma = \to = \blacktriangleright \cup \triangleright$ and $\blacktriangleright, \triangleright$ jumping, $\gamma \nearrow\!\!\!\nearrow \hat{\gamma}$ for some $\hat{\gamma}$.*

Proof. With the above replacements, the proof *is* that of Lem. 4 including it being illustrated by Fig. 1, where $\not\downarrow$ now marks $\blacktriangleright$-normal forms *along* γ, not necessarily *in* γ, *cf.* [5, Fig. 7]. ◀

▶ **Example 11.** $\blacktriangleright, \triangleright$ given by $a' \triangleright a$ and $\gamma : a \triangleright b \blacktriangleright c$ and $\epsilon : a \blacktriangleright a' \blacktriangleright a' \blacktriangleright \dots$, is jumping but not affluent: γ promotes (only) to ϵ with objects of γ all *along* γ, not all *in* γ (a' isn't).

▶ **Corollary 12.** *Let $\to := \blacktriangleright \cup \triangleright$.* **1.** *$\blacktriangleright, \triangleright$ are terminating iff $\to$ is, for jumping $\blacktriangleright, \triangleright$ [2];* **2.** *$a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright^{\omega}$ if a is $\blacktriangleright$-terminating but not $\to$-terminating, and $\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright$ ($\diamond$) [4, Lem. 51].*

Proof. 2. If $a \to^{\omega} \nearrow\!\!\!\nearrow \hat{\gamma}$ then $\hat{\gamma}$ is not of shape $a \blacktriangleright\!\!\blacktriangleright \cdot \triangleright\!\!\triangleright \cdot \blacktriangleright^{\omega}$ as $\diamond$ entails $\triangleright\!\!\triangleright \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleright\!\!\triangleright$. ◀

Call a family $\mathcal{F}$ *jumping* [1] if $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ for $i \in I$.[5]

▶ **Theorem 13.** *Let γ be a reduction with $\to|\gamma = \to = \bigcup \mathcal{F}$ with $\mathcal{F} := (\to)_{[\ell,n]}$ and let $\nearrow$ be defined as in Def. 3 under these assumptions. If $\mathcal{F}$ is jumping, then $\gamma \nearrow \epsilon$ for some ϵ.*

Proof. With the above replacements, the proof *is* that of Thm. 7, using Lem. 10 instead of 4 with the IH applicable since $(\to|\delta)_J$-steps compose to other such by J being a *suffix* of I. ◀

▶ **Corollary 14.** *Let $\mathcal{F} := (\to)_I$, $\to := \bigcup \mathcal{F}$.* **1.** *$\mathcal{F}$ is terminating iff $\to$ is, for jumping $\mathcal{F}$ [1];* **2.** *$\mathcal{F} = (\blacktriangleright, \triangleright, \gg)$ is terminating iff $\to$ is, if $(\triangleright \cup \gg) \cdot \blacktriangleright \subseteq (\triangleright \cup \gg) \cup (\blacktriangleright \cdot \to^*)$ and $\gg \cdot \triangleright \subseteq \gg \cup (\triangleright \cdot (\triangleright \cup \gg)^*)$ [1, Thm. 8].*

4 Blending Affine and Jumping Families

Blending families [1] is limited only by (correctnes of) one's illusion. We give examples.

▶ **Lemma 15.** *Let γ be an $\mathcal{F}$-reduction for $\mathcal{F} := (\to)_I$ and $I := [\ell, n]$, and let $\to := \bigcup \mathcal{F}$. $\gamma \nearrow \epsilon$ for some ϵ, if:* **1.** *$\to_{>i} \cdot \to_i \subseteq \to_{>\ell} \cup (\to_\ell \cdot \twoheadrightarrow)$ for $\ell \leq i \leq n$* (affluence$_\frown$)*; or* **2.** *$\blacktriangleright_i, \to_{>i}$ is affluent for all i* (partite)*, for ϵ a $\mathcal{G}$-reduction ϵ,*[6] *with $\mathcal{G} := (\blacktriangleright)_{[\ell,n]}$ and $\blacktriangleright_i := \to_i^+$ for $i < n$ and $\blacktriangleright_n := \to_n$; or* **3.** *$\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup \to_i^+ \cup (\to_\ell \cdot \twoheadrightarrow)$ for $\ell \leq i < n$* (partite$_\frown$)*.*

Proof. 1. γ can be seen as a $(\to_\ell|\gamma, \to_{>\ell}|\gamma)$-reduction. By assumption and Lem. 10, $\gamma \nearrow\!\!\!\nearrow \hat{\gamma}$ for some $\hat{\gamma}$. Let δ be the $\to_{>\ell}|\gamma$-subreduction of $\hat{\gamma}$. Viewed as a $(\to_{>\ell}|\gamma)\lceil\delta$-reduction, Thm. 7 yields $\delta \nearrow \hat{\delta}$ for some $\hat{\delta}$, and we conclude by setting ϵ to $\hat{\gamma}$ in which δ is replaced by $\hat{\delta}$. Thm. 7 is applicable to δ since the second disjunct of affluence$_\frown$ cannot hold, reducing it to affluence: if $a \to_\ell c \twoheadrightarrow b$ for a, b in δ, with a the source of some $(\to_{>\ell}|\gamma)$-step in δ, then a, b and hence c would be along γ by δ being part of $\hat{\gamma}$, contradicting that a be in $\to_\ell|\gamma$-normal form per $\gamma \nearrow\!\!\!\nearrow \hat{\gamma}$. **2.** Adaptating the proof of Thm. 7 using *bait and switch* in the induction: a $\to_{\geq i}$-reduction γ (the bait) can be seen as a $(\blacktriangleright_i\lceil\gamma, \to_{>i}\lceil\gamma)$-reduction (the switch) since $\to_i \subseteq \to_i^+$. By assumption and Lem. 4 that promotes to some $\hat{\gamma}$, from which we conclude by applying the IH to its $\to_{>i}\lceil\gamma$-subreduction, yielding ϵ; **3.** As for 1 but using 2 instead of Thm. 7. ◀

[5] We based jumping of $(\to)_{[\ell,n]}$ on that of $(\to)_{[\ell+1,n]}$. For basing it on $(\to)_{[\ell,n\dot{-}1]}$ see [1, Thm. 7, Cor. 20].
[6] Though objects in ϵ must be in γ, this no longer holds if we unfold its $\to_i^+$-steps into single $\to_i$-steps.

▶ **Corollary 16.** *$\mathcal{F} := (\to)_I$ is terminating iff $\to := \bigcup \mathcal{F}$ is, if:* **1.** *affluence$_{\curvearrowright}$; or* **2.** *partite; or* **3.** *partite$_{\curvearrowright}$ [1, Thm. 22]; or* **4.** *$\mathcal{F} = (\blacktriangleright, \triangleright, \gg)$ and $(\triangleright \cup \gg) \cdot \blacktriangleright \subseteq \triangleright \cup \gg \cup (\blacktriangleright \cdot (\blacktriangleright \cup \triangleright \cup \gg)^*)$ and $\gg \cdot \triangleright \subseteq \gg \cup \triangleright^+ \cup (\blacktriangleright \cdot (\blacktriangleright \cup \triangleright \cup \gg)^*)$ (jumping$_3$) [1, Thm. 4]; or* **5.** *for some $k \leq n$, $\to_{>i} \cdot \to_i \subseteq \to_{>0} \cup (\to_0 \cdot \twoheadrightarrow)$ for $i < k$, and $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ for $k \leq i < n$ (affluence$^{\curvearrowleft}_{\curvearrowright}$); or* **6.** *for some $k \leq n$, $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup \to_i^+ \cup (\to_0 \cdot \twoheadrightarrow)$ for $i < k$, and $\to_{>i} \cdot \to_i \subseteq \to_{>i} \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ for $k \leq i < n$ (partite$^{\curvearrowleft}_{\curvearrowright}$) [1, Thm. 28].*

▶ **Example 17.** $\to_{>i} \cdot \to_i \subseteq \to \cup (\to_i \cdot \twoheadrightarrow_{\geq i})$ for all i, may seem a harmless blend of affluence and jumping, but though it holds for the terminating family $\mathcal{F} := (\blacktriangleright, \triangleright, \gg)$ [1, Ex. 9(a)] given by $b \blacktriangleright d$, $c \triangleright d \triangleright a \triangleright b$, and $a \gg d$, $b \gg c$, $\bigcup \mathcal{F}$ is non-terminating: $a \gg d \triangleright a$.[7]

5 Conclusions

By modularising (factoring through promotion Def. 3) and unifying (transitivity, affluence, jumping via restriction (Defs. 2 and 9)) proofs, we related disjointed results on disjunctive termination [3, 2, 6, 1] and improved upon them (*e.g.*, Cor. 8(1)). Ideas for further research: (I) Recast using 2-*rewriting* for *transducing* (infinite) reductions; (II) Exploit progressiveness and sharpen promotion to (re)gain *quantitative* results, like [7] and *quasi-commutation*; (III) Automate results in *tools* (to handle, *e.g.*, Ex. 18) and *formalise* them (*axiomatically?*).

▶ **Example 18.** **1.** [6, Fig. 2 (**CHOICE**)] presents a program having transition relation R given by relating pairs of natural numbers $\langle x, y \rangle$ and $\langle x', y' \rangle$ if the latter is either $\langle x \dot{-} 1, x \rangle$ or $\langle y \dot{-} 2, x + 1 \rangle$, assuming $x, y > 0$, and a doubleton family $\mathcal{F} := (\blacktriangleright, \triangleright)$ of relations $\blacktriangleright := \neg P(x, y) \wedge (Q \vee P(x', y'))$ and $\triangleright := P(x, y) \wedge Q \wedge P(x', y')$ for $Q := x + y > x' + y'$ and $P(n, m) := m \dot{-} 2 \leq n \leq m \dot{-} 1$. Then $R \subseteq \to := \bigcup \mathcal{F}$, $\triangleright \cdot \blacktriangleright = \emptyset$, and $\mathcal{F}$ is terminating since Q is and since P and $\neg P$ do not compose, yielding affluence of $\blacktriangleright, \triangleright$ hence termination of $\to$ by Cor. 5(1), so R is terminating. **2.** [7, Ex. 6.1] presents a program having transition relation R given by $\langle x, y \rangle \to_{x>y \& x>0 \& y>0} \langle y, 2^{x+y} \rangle$ and $\langle x, y \rangle \to_{\neg(x>y) \& x>0 \& y>0} \langle x, y - 1 \rangle$, and a doubleton family $\mathcal{F} := (\blacktriangleright, \triangleright)$ of terminating relations $\blacktriangleright := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid x > 0 \& x > x'\}$ and $\triangleright := \{(\langle x, y \rangle, \langle x', y' \rangle) \mid y > 0 \,\&\, y > y'\}$. Then $R \subseteq \bigcup \mathcal{F}$ but affluence of $\mathcal{F}$ fails. Restricting $\triangleright$ by $\{(\langle x, y \rangle, \langle x', y' \rangle) \mid x \geq x'\}$ guided by R, both hold and Cor. 5(1) applies.

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[7] Such (very) finite counterexamples can be found automatically, *e.g.*, by Zantema's tool Carpa.