Modularity of Termination in Probabilistic Term Rewriting

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Abstract

We investigate the modularity of probabilistic notions of termination in term rewriting. In the probabilistic setting, there are several interesting termination properties: Almost-sure termination (termination with probability 1), positive almost-sure termination (finite expected runtime of each rewrite sequence), and strong almost-sure termination (expected runtime is bounded by a constant for each start term). A property is called *modular* if it is preserved for certain unions of probabilistic term rewrite systems. We show that these three termination properties have different modularity behavior for innermost rewriting. Utilizing known relations between innermost and full probabilistic rewriting allows us to obtain modularity results for full probabilistic rewriting as well.

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Related Version *See [8]. Full version, including all proofs*: `https://arxiv.org/abs/2409.17714`

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1 Probabilistic Term Rewriting

We assume familiarity with non-probabilistic term rewriting [2] and briefly recapitulate probabilistic term rewriting, see, e.g., [1, 3, 6]. We write $\mathcal{T}(\Sigma, \mathcal{V})$ for the set of all *terms* over a (possibly infinite) countable set of *function symbols* $\Sigma = \biguplus_{k \in \mathbb{N}} \Sigma_k$ and a (possibly infinite) countable set of *variables* $\mathcal{V}$. In contrast to ordinary term rewrite systems (TRSs), a probabilistic term rewrite system (PTRS) has finite multi-distributions on the right-hand sides of its rewrite rules. A finite *multi-distribution* μ on $\mathcal{T}(\Sigma, \mathcal{V})$ is a finite multiset of pairs $(p : t)$, where $0 < p \leq 1$ is a probability and $t \in \mathcal{T}(\Sigma, \mathcal{V})$, such that $\sum_{(p:t) \in \mu} p = 1$. $\mathrm{FDist}(\mathcal{T}(\Sigma, \mathcal{V}))$ is the set of all finite multi-distributions on $\mathcal{T}(\Sigma, \mathcal{V})$. For $\mu \in \mathrm{FDist}(\mathcal{T}(\Sigma, \mathcal{V}))$, its *support* is the multiset $\mathrm{Supp}(\mu) = \{t \mid (p : t) \in \mu \text{ for some } p\}$. A *probabilistic rewrite rule* is a pair $(\ell \to \mu) \in \mathcal{T}(\Sigma, \mathcal{V}) \times \mathrm{FDist}(\mathcal{T}(\Sigma, \mathcal{V}))$ such that $\ell \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(\ell)$ for every $r \in \mathrm{Supp}(\mu)$. A *probabilistic term rewrite system* is a (possibly infinite) countable set $\mathcal{P}$ of probabilistic rewrite rules. Similar to TRSs, the PTRS $\mathcal{P}$ induces a *rewrite relation* $\xrightarrow{\mathbf{f}}_{\mathcal{P}} \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathrm{FDist}(\mathcal{T}(\Sigma, \mathcal{V}))$ where $s \xrightarrow{\mathbf{f}}_{\mathcal{P}} \{p_1 : t_1, \ldots, p_k : t_k\}$ if there is a position π, a rule $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{P}$, and a substitution σ such that $s|_\pi = \ell\sigma$ and $t_j = s[r_j\sigma]_\pi$ for all $1 \leq j \leq k$. Here, $\mathbf{f}$ stands for "full rewriting".[1] We call $s \xrightarrow{\mathbf{f}}_{\mathcal{P}} \mu$ an *innermost* rewrite step (denoted $s \xrightarrow{\mathbf{i}}_{\mathcal{P}} \mu$) if all proper subterms of the used redex $\ell\sigma$ are in normal form w.r.t. $\mathcal{P}$. For example, the PTRS $\mathcal{P}_{\mathsf{rw}}$ with the only rule $\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ 1/2 : \mathsf{0}\}$ corresponds to a symmetric random walk on the number of g-symbols in a term.

[1] In the literature, one usually simply writes $\to_{\mathcal{P}}$ instead. Moreover, a "full" strategy sometimes refers to a specific strategy different from $\to_{\mathcal{P}}$, such as *full substitution rewriting* in Def. 4.9.5 (v) of [9]. We use $\xrightarrow{\mathbf{f}}_{\mathcal{P}} = \to_{\mathcal{P}}$ here to clearly distinguish between $\xrightarrow{\mathbf{f}}_{\mathcal{P}}$ and $\xrightarrow{\mathbf{i}}_{\mathcal{P}}$ in the following.

Next, we recapitulate the different notions of termination for PTRSs, see [1, 3, 6]. Let $\to \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathrm{FDist}(\mathcal{T}(\Sigma, \mathcal{V}))$ be an arbitrary *probabilistic relation*, e.g., $\to = \xrightarrow{\mathtt{f}}_{\mathcal{P}}$ or $\to = \xrightarrow{\mathtt{i}}_{\mathcal{P}}$ for a PTRS $\mathcal{P}$, and let $\mathtt{NF}_{\to}$ be the set of all normal forms for $\to$. We track all possible rewrite sequences with their corresponding probabilities by lifting $\to$ to *rewrite sequence trees (RSTs)* [6]. The nodes v of an $\to$-RST are labeled by pairs $(p_v : t_v)$ of a probability p_v and a term t_v, where the root is always labeled with the probability 1. For each node v with the successors $w_1, \ldots, w_k$, the edge relation represents a step with the relation $\to$, i.e., $t_v \to \{\frac{p_{w_1}}{p_v} : t_{w_1}, \ldots, \frac{p_{w_k}}{p_v} : t_{w_k}\}$. For an $\to$-RST $\mathfrak{T}$, let $N^{\mathfrak{T}}$ denote the set of its nodes and $\mathrm{Leaf}^{\mathfrak{T}}$ denote the set of its leaves. For example, Fig. 1 depicts an $\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}$-RST, which is also an $\xrightarrow{\mathtt{i}}_{\mathcal{P}_{\mathsf{rw}}}$-RST as it only uses innermost rewrite steps. As usual in term rewriting, a term might contain several redexes and several rules might be applicable to each redex. Thus, there can be several $\to$-RSTs with the same root.

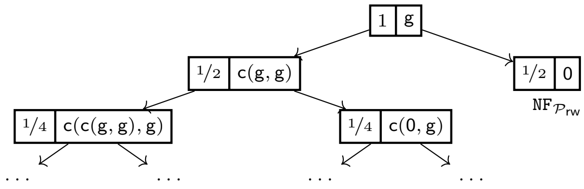

Figure 1 $\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}$-RST which only uses innermost steps.

To express the concept of almost-sure termination, we have to determine the probabilities of the leaves of RSTs. While we define our notions of termination via RSTs, they are equivalent to the ones in [1, 3] where termination is defined via a lifting of $\to$ to multisets or via stochastic processes.

▶ **Definition 1** (Almost-Sure Termination). *For any $\to$-RST $\mathfrak{T}$ we define its* termination probability $|\mathfrak{T}| = \sum_{v \in \mathrm{Leaf}^{\mathfrak{T}}} p_v$. *Then* $\mathtt{AST}_{\to}$ *holds if for all $\to$-RSTs $\mathfrak{T}$ we have $|\mathfrak{T}| = 1$.*

▶ **Example 2.** For every extension $\mathfrak{T}$ of the $\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}$-RST in Fig. 1, we have $|\mathfrak{T}| = 1$. Indeed, we have $\mathtt{AST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}}$ and thus also $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}_{\mathsf{rw}}}}$.

Next, we define *positive* almost-sure termination, which considers the *expected derivation length* $\mathrm{edl}(\mathfrak{T})$ of an RST $\mathfrak{T}$, i.e., the expected number of steps until one reaches a normal form. For positive almost-sure termination, we require that the expected derivation length of every possible rewrite sequence is finite.

▶ **Definition 3** (Positive Almost-Sure Termination, edl). *We define the* expected derivation length *of an $\to$-RST $\mathfrak{T}$ to be $\mathrm{edl}(\mathfrak{T}) = \sum_{v \in N^{\mathfrak{T}} \setminus \mathrm{Leaf}^{\mathfrak{T}}} p_v$. Then, we have* $\mathtt{PAST}_{\to}$ *if $\mathrm{edl}(\mathfrak{T})$ is finite for every $\to$-RST $\mathfrak{T}$.*

▶ **Example 4.** The expected derivation length $\mathrm{edl}(\mathfrak{T})$ is infinite for every infinite innermost extension $\mathfrak{T}$ of the $\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}$-RST in Fig. 1 such that for every leaf $v \in \mathrm{Leaf}^{\mathfrak{T}}$ the corresponding term t_v is a normal form. Hence, $\mathtt{PAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}_{\mathsf{rw}}}}$ does not hold, and $\mathtt{PAST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}_{\mathsf{rw}}}}$ does not hold either.

Finally, we define *strong almost-sure termination* [1, 4], which is even stricter than $\mathtt{PAST}$ in case of non-determinism. It requires a finite bound on the expected derivation lengths of all rewrite sequences with the same start term. For a term $t \in \mathcal{T}(\Sigma, \mathcal{V})$, the *expected derivation height* $\mathrm{edh}_{\to}(t)$ considers all RSTs that start with t.

▶ **Definition 5** (Strong Almost-Sure Termination, edh). *We have* $\mathtt{SAST}_{\to}$ *if* $\mathrm{edh}_{\to}(t) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T}$ *is an $\to$-RST whose root is labeled with $(1 : t)\}$ is finite for all $t \in \mathcal{T}(\Sigma, \mathcal{V})$.*

2 Modularity

For a PTRS $\mathcal{P}$, we decompose its signature $\Sigma = \Sigma_C \uplus \Sigma_D$ such that $f \in \Sigma_D$ if $f = \mathrm{root}(\ell)$ for some rule $\ell \to \mu \in \mathcal{P}$. The symbols in Σ_C and Σ_D are called *constructors* and *defined* *symbols*, respectively. To distinguish the functions symbols of different PTRSs $\mathcal{P}$, in the following we write $\Sigma_D^{\mathcal{P}}$, $\Sigma_C^{\mathcal{P}}$, and $\Sigma^{\mathcal{P}}$ for the defined symbols, constructor symbols, and all function symbols occurring in the rules of $\mathcal{P}$, respectively.

We study two different forms of unions, namely *disjoint unions* (Sect. 2.1), where both PTRSs do not share any function symbols, and *shared constructor unions* of PTRSs (Sect. 2.2) which may have common constructor symbols, but whose defined symbols are disjoint. In both cases, we restrict ourselves to innermost rewriting, as ordinary termination of TRSs for full rewriting is already not modular. One can lift our results on innermost rewriting to full rewriting for those classes of PTRSs $\mathcal{P}$ where, e.g., $\mathtt{AST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}}} = \mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$, see [7, 8]. Finally in Sect. 2.3, we investigate the preservation of both full and innermost probabilistic rewriting under signature extensions.

2.1 Disjoint Unions

We first consider unions of systems that do not share any function symbols, i.e., we consider two PTRSs $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ such that $\Sigma^{\mathcal{P}^{(1)}} \cap \Sigma^{\mathcal{P}^{(2)}} = \varnothing$. In the non-probabilistic setting, [5] showed that innermost termination is modular for disjoint unions. This result can be lifted to $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$ and $\mathtt{SAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$, but it does not hold for $\mathtt{PAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$. We first investigate $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$, as illustrated by the following example.

▶ **Example 6.** Consider the PTRS $\mathcal{P}_1 = \mathcal{P}_1^{(1)} \cup \mathcal{P}_1^{(2)}$ given by

$$\mathcal{P}_1^{(1)} : \mathsf{f}(x) \to \{1/2 : \mathsf{f}(x), 1/2 : \mathsf{a}\} \qquad \mathcal{P}_1^{(2)} : \mathsf{g}(x) \to \{1/2 : \mathsf{g}(x), 1/2 : \mathsf{b}\}$$

$\mathcal{P}_1^{(1)}$ and $\mathcal{P}_1^{(2)}$ both correspond to a fair coin flip, where one terminates when obtaining heads. Hence, for both systems we have $\mathtt{AST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}_1^{(1)}}}$ and $\mathtt{AST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}_1^{(2)}}}$, and thus also $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}_1^{(1)}}}$ and $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}_1^{(2)}}}$. Furthermore, $\Sigma^{\mathcal{P}_1^{(1)}} \cap \Sigma^{\mathcal{P}_1^{(2)}} = \varnothing$, i.e., $\mathcal{P}_1$ is a disjoint union. When reducing a term like $\mathsf{f}(\mathsf{g}(x))$ which contains symbols from both systems, then we first reduce the innermost redex $\mathsf{g}(x)$ until we reach a normal form. Due to the innermost strategy, we cannot rewrite at the position of f beforehand. This reduction only uses one of the two systems, namely $\mathcal{P}_1^{(2)}$, hence it terminates with probability 1. Then, we use the next innermost redex, which will be $\mathsf{f}(\mathsf{b})$, using only rules of $\mathcal{P}_1^{(1)}$ until we reach a normal form, where symbols from $\Sigma^{\mathcal{P}_1^{(2)}}$ do not influence the reduction. The reason is that all subterms below or at positions of symbols from $\Sigma^{\mathcal{P}_1^{(2)}}$ are in normal form, and there is no symbol from $\Sigma^{\mathcal{P}_1^{(2)}}$ above the f at the root position. Again, this reduction terminates with probability 1. Thus, in the end, our reduction starting with $\mathsf{f}(\mathsf{g}(x))$ also terminates with probability 1, and the same holds for arbitrary start terms, which implies $\mathtt{AST}_{\xrightarrow{\mathtt{f}}_{\mathcal{P}_1}}$.

In Ex. 6, we considered the term $\mathsf{f}(\mathsf{g}(x))$ where we swap once between a symbol f from $\Sigma^{\mathcal{P}_1^{(1)}}$ and a symbol g from $\Sigma^{\mathcal{P}_1^{(2)}}$ on the path from the root to the "leaf" of the term. In the proof of Thm. 7 we lift the argumentation of Ex. 6 to arbitrary terms via induction. This proof idea was also used by [5] in the non-probabilistic setting to show the modularity of innermost termination for disjoint unions of TRSs.

▶ **Theorem 7** (Modularity of $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$ for Disjoint Unions). *Let $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ be PTRSs with $\Sigma^{\mathcal{P}^{(1)}} \cap \Sigma^{\mathcal{P}^{(2)}} = \varnothing$. Then we have:* $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}}} \iff \mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}^{(1)}}}$ *and* $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}^{(2)}}}$.

In contrast to $\mathtt{AST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$, $\mathtt{PAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$ cannot be modular due to the potential extension of the signature (see Thm. 13, which shows that $\mathtt{PAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$ is not closed under signature extensions).

Finally, we consider $\mathtt{SAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$. To prove that $\mathtt{SAST}_{\xrightarrow{\mathtt{i}}_{\mathcal{P}}}$ is modular for disjoint unions, we have to show that the expected derivation height of any term t is finite. However, after rewriting t's proper subterms to normal forms, as we did in Ex. 6 and in the induction proof of Thm. 7, we may end up with infinitely many different terms. All their expected derivation heights have to be considered in order to compute the expected derivation height of t.

▶ **Example 8.** Consider the PTRSs $\mathcal{P}_2^{(1)}$ and $\mathcal{P}_2^{(2)}$ with

$$\mathcal{P}_2^{(1)} : \mathsf{f}(\mathsf{s}(x), y) \to \{1 : \mathsf{f}(x, y)\} \qquad \mathcal{P}_2^{(2)} : \mathsf{g}(x) \to \{1 : x\}$$
$$\mathsf{f}(x, \mathsf{s}(y)) \to \{1 : \mathsf{f}(x, y)\}$$
$$\mathsf{a} \to \{1/2 : \mathsf{0}, 1/2 : \mathsf{s}(\mathsf{a})\}$$

Clearly, we have both $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}_2^{(1)}}}$ and $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}_2^{(2)}}}$. Now consider the term $t = \mathsf{f}(\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a}))$. Due to the innermost strategy, we have to rewrite its proper subterms first. When proceeding in a similar way as in the induction proof of Thm. 7, then one would first construct bounds on the expected derivation heights of the proper subterms, and then use them to obtain a bound on the expected derivation height of the whole term t. However, reducing t's proper subterms can create infinitely many different terms, i.e., all terms of the form $\mathsf{f}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}^m(\mathsf{0}))$ for any $n, m \in \mathbb{N}$ can be reached with a certain probability. Since there is no finite supremum on the derivation height of $\mathsf{f}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}^m(\mathsf{0}))$ for all $n, m \in \mathbb{N}$, one would have to take the individual probabilities for reaching the terms $\mathsf{f}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}^m(\mathsf{0}))$ into account in order to prove that the expected derivation height of t is indeed finite.

As shown in Ex. 8, there may be infinitely many terms t' after an inductive step, e.g., in Ex. 8, t' can be any term of the form $\mathsf{f}(\mathsf{s}^n(\mathsf{0}), \mathsf{s}^m(\mathsf{0}))$. However, this infinite set of terms can be over-approximated by a finite number of representatives (with finite expected derivation height), leading to the following result.

▶ **Theorem 9** (Modularity of $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}}}$ for Disjoint Unions)**.** *Let $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ be PTRSs with $\Sigma^{\mathcal{P}^{(1)}} \cap \Sigma^{\mathcal{P}^{(2)}} = \varnothing$. Then we have: $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}}} \iff \mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}^{(1)}}}$ and $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}^{(2)}}}$.*

2.2 Shared Constructor Unions

Now we consider unions of PTRSs that may share constructor symbols, i.e., we consider two PTRSs $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ such that $\Sigma_D^{\mathcal{P}^{(1)}} \cap \Sigma_D^{\mathcal{P}^{(2)}} = \varnothing$, called *shared constructor unions*.

In the non-probabilistic setting, innermost termination is also modular for shared constructor unions [5]. However, $\mathtt{PAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}}}$ was already not modular w.r.t. disjoint unions, so this also holds for shared constructor unions. Moreover, $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}}}$ also turns out to be not modular anymore for shared constructor unions.

Counterexample 10. Consider the PTRS $\mathcal{P}_3 = \mathcal{P}_3^{(1)} \cup \mathcal{P}_3^{(2)}$ with the rules

$$\mathcal{P}_3^{(1)} : \mathsf{f}(\mathsf{c}(x, y)) \to \{1 : \mathsf{c}(\mathsf{f}(x), \mathsf{f}(y))\}$$
$$\mathsf{f}(\mathsf{0}) \to \{1 : \mathsf{0}\}$$
$$\mathcal{P}_3^{(2)} : \mathsf{g}(x) \to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 1/2 : x\}$$
$$\mathsf{d}(x) \to \{1 : \mathsf{c}(x, x)\}$$

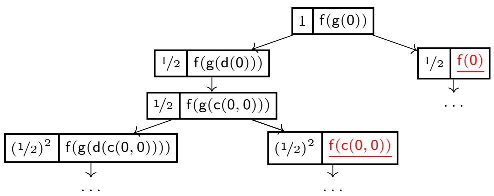

Figure 2 Infinite $\xrightarrow{\mathrm{i}}_{\mathcal{P}_3}$-RST.

While $\mathcal{P}_3^{(1)}$ and $\mathcal{P}_3^{(2)}$ do not have any common defined symbols, they share the constructor c. We do not have $\mathtt{PAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}_3}}$ (and thus, not $\mathtt{SAST}_{\xrightarrow{\mathrm{i}}_{\mathcal{P}_3}}$ either), as the infinite $\xrightarrow{\mathrm{i}}_{\mathcal{P}_3}$-RST depicted in Fig. 2 has an infinite expected derivation length. For any $n \in \mathbb{N}$, each red underlined term $\mathsf{f}(\mathsf{c}^n(\mathsf{0}, \mathsf{0}))$ in the tree above can start a reduction of at least length 2^n, where $\mathsf{c}^n(\mathsf{0}, \mathsf{0})$ corresponds to the full binary tree of height n with c in inner nodes and 0 in the leaves. Hence, the term $\mathsf{f}(\mathsf{g}(0))$ has an expected derivation height of at least $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot 2^n = \sum_{n=0}^{\infty} \frac{1}{2}$, which diverges to infinity.

On the other hand, we have $\mathtt{SAST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}_3^{(1)}}}$, as $\mathcal{P}_3^{(1)}$ is a PTRS with only trivial probabilities that corresponds to a terminating TRS. Moreover, $\mathtt{SAST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}_3^{(2)}}}$ holds as well, as the d-rule can increase the number of c-symbols in a term exponentially, but those c-symbols will never be used. Thus, $\mathtt{SAST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}}}$ is not modular for shared constructor unions.

For $\mathtt{AST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}}}$ we obtain a similar result for shared constructor unions as for disjoint unions.

▶ **Theorem 11** (Modularity of $\mathtt{AST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}}}$ for Shared Constructor Unions). *Let $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ be PTRSs with $\Sigma_D^{\mathcal{P}^{(1)}} \cap \Sigma_D^{\mathcal{P}^{(2)}} = \varnothing$. Then we have: $\mathtt{AST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}}} \iff \mathtt{AST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}^{(1)}}}$ and $\mathtt{AST}_{\xrightarrow{\mathbf{i}}_{\mathcal{P}^{(2)}}}$.*

2.3 Signature Extensions

Finally, we consider signature extensions. The following theorem shows that both $\mathtt{AST}$ and $\mathtt{SAST}$ are closed under extensions of the signature, for both innermost and full rewriting.

▶ **Theorem 12** (Signature Extensions for $\mathtt{AST}_{\xrightarrow{s}_{\mathcal{P}}}$ and $\mathtt{SAST}_{\xrightarrow{s}_{\mathcal{P}}}$). *Let $\mathcal{P}$ be a PTRS, $s \in \{\mathbf{f}, \mathbf{i}\}$, and let Σ' be some signature. Then we have: $\mathtt{AST}_{\xrightarrow{s}_{\mathcal{P}}}$ over $\Sigma^{\mathcal{P}}$ $\iff$ $\mathtt{AST}_{\xrightarrow{s}_{\mathcal{P}}}$ over $\Sigma^{\mathcal{P}} \cup \Sigma'$ and $\mathtt{SAST}_{\xrightarrow{s}_{\mathcal{P}}}$ over $\Sigma^{\mathcal{P}}$ $\iff$ $\mathtt{SAST}_{\xrightarrow{s}_{\mathcal{P}}}$ over $\Sigma^{\mathcal{P}} \cup \Sigma'$.*

However, $\mathtt{PAST}$ is not closed under signature extensions.

▶ **Theorem 13** (Signature Extensions for $\mathtt{PAST}_{\xrightarrow{s}_{\mathcal{P}}}$). *Let $s \in \{\mathbf{f}, \mathbf{i}\}$. There exists a PTRS $\mathcal{P}$ and signatures Σ, Σ' with $\Sigma \subset \Sigma'$ such that $\mathtt{PAST}_{\xrightarrow{s}_{\mathcal{P}}}$ holds over Σ, but not over Σ'.*

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