Termination of String Rewriting with Matrix Interpretations

Method and Implementations

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String Rewriting

Why study string rewriting?

- Oriented equations
  $\rightsquigarrow$ (semi-) group theory
- Universal computation model
  $\rightsquigarrow$ recursion / complexity theory
- Model for non-deterministic processes
- Prototype for more general rewriting systems:
  term / higher-order / graph / ...

String Rewriting: Definitions

- Letter: element of a set Σ, the alphabet
- String: sequence of letters. Σ^* is the set of strings over Σ
- String rewriting system: set of rules of the form $\ell \to r$,
  i.e. a set $R \subseteq \Sigma^* \times \Sigma^*$
- Rewrite step: replace the left hand side of rule $\ell \to r$ by its right hand side: $x\ell y \to_R xry$ within context $x, y \in \Sigma^*$
- Derivation: chain of rewrite steps

String Rewriting: Example

$R = \{aa \to bc,\ bb \to ac,\ cc \to ab\}$ induces derivation

$$
\begin{aligned}
&b\,b\,\boxed{a\,a} \to_R \\
&b\,\boxed{b\,b}\,c \to_R \\
&b\,a\,\boxed{c\,c} \to_R \\
&b\,\boxed{a\,a}\,b \to_R \\
&\boxed{b\,b}\,c\,b \to_R \\
&a\,\boxed{c\,c}\,b \to_R \\
&a\,a\,b\,b \to_R \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Is there an infinite derivation?
No: Termination proof by a matrix interpretation.
Exponential upper bound on dc_R.
Open problem: polytime upper bound?

Termination

Why study termination? Well ...

Definition: System R is terminating
if any R-derivation contains only finitely many steps.

- Notation $\mathrm{SN}(R)$: R is *strongly normalizing*
- In other words: $\to_R$ is well-founded.

Expl.s of terminating systems:

- $\{aab \to ba\}$
- $\{ab \to ba\}$
- $\{ab \to baa\}$
- $\{aa \to aba\}$

Zantema's System $\{aabb \to bbbaaa\}$

A test case for automated termination methods (z001).

$$
a \mapsto \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}
\qquad
b \mapsto \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}
$$

$$
(\ell \to r) \mapsto \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

- This interpretation proves termination since *all* entries are ≥ 0 and *marked* entries are $\boxed{\geq 1}$
- Found automatically / Underlying theory elementary / Fast verification

Relative Termination

allows to remove rules successively $\rightsquigarrow$
Modular termination proofs

Definition: System R is terminating relative to system S
if any $R \cup S$-derivation contains only finitely many R-steps.

- Notation: $\mathrm{SN}(R/S)$
- In other words: $\to_S^* \circ \to_R \circ \to_S^*$ is well-founded.

$\boxed{\mathrm{SN}(R/S) \text{ and } \mathrm{SN}(S) \text{ imply } \mathrm{SN}(R \cup S)}$

Expl: $\{aa \to aba\}$ is terminating relative to $\{b \to bb\}$.

Termination via Interpretations

Interpretations as order preserving mappings
into well-founded domains:

- Let R and S be rewriting systems over Σ^*.
- Let $(N, \geq)$ be a well-founded partial order.

If a mapping $i : \Sigma^* \to N$ is order preserving both

- from $(\Sigma^*, \to_R)$ to $(N, >)$ and
- from $(\Sigma^*, \to_S)$ to $(N, \geq)$,

then R is terminating relative to S.

$$
\begin{tikzcd}
\bullet \arrow[r, "*"', "S"] \arrow[d] & \bullet \arrow[r, "R"] \arrow[d] & \bullet \arrow[r, "*"', "S"] \arrow[d] & \bullet \arrow[r, "R"] \arrow[d] & \bullet \arrow[r, "*"', "S"] \arrow[d] & \bullet \arrow[r, dotted] \arrow[d] & {} \\
\bullet \arrow[r, "\geq"] & \bullet \arrow[r, ">"] & \bullet \arrow[r, "\geq"] & \bullet \arrow[r, ">"] & \bullet \arrow[r, "\geq"] & \bullet \arrow[r, dotted] & {}
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$$

Ring Interpretations

Interpret the free monoid of strings in a ring:

- concatenation of factors $\mapsto$ multiplication
- replacement of factors $\mapsto$ substraction

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For termination: Use an (infinite) ordered ring, which is well-founded (on its "positive cone").

- Expl: $(\mathbb{Z}, 0, 1, +, \cdot)$ works for $\{aab \to ba\}$, but doesn't work for $\{ab \to ba\}$ as multiplication is commutative.

$\leadsto$ Use a $\boxed{\text{non-commutative ring}}$, e.g., a $\boxed{\text{matrix ring}}$

Well-founded Rings

A *partially ordered ring* $(D, 0, 1, +, \cdot, \geq)$:

- $(D, 0, +)$ an Abelian group, $(D, 1, \cdot)$ a monoid.
- Multiplication distributes over addition from both sides. (Multiplication not necessarily commutative / invertible.)
- $\geq$ is a compatible partial order:

$$a \geq b \Rightarrow a + c \geq b + c$$
$$a \geq b \wedge c \geq 0 \Rightarrow a \cdot c \geq b \cdot c \wedge c \cdot a \geq c \cdot b$$

Its *positive cone*: $N = \{d \in D \mid d \geq 0\}$,
its *strictly positive cone*: $P = N \setminus \{0\} = \{d \in D \mid d > 0\}$.
The ring is *well-founded* if $>$ is well-founded on N.

- Note: The order is uniquely determined by these cones: $a \geq b$ iff $a - b \in N$ and $a > b$ iff $a - b \in P$.
- Note: $N \cdot N \subseteq N$, but $P \cdot P \not\subseteq P$ if zero divisors exist.

Ring Interpretations (cont'd)

A *ring interpretation* of alphabet Σ is a mapping $i : \Sigma \to D$

- extended to a mapping $i : \Sigma^* \to D$ on strings by

$$\boxed{i(s_1 \cdot \ldots \cdot s_n) = i(s_1) \cdot \ldots \cdot i(s_n)}$$

- extend to a mapping $i : \Sigma^* \times \Sigma^* \to D$ on rules by

$$\boxed{i(\ell \to r) = i(\ell) - i(r)}$$

Termination via Ring Interpretations

Apply ring interpretations for proving termination:
Ensure $i(x\ell y) > i(xry)$ for each step $x\ell y \to_R xry$, i.e.,

$$i(x\ell y) - i(xry) = i(x)i(\ell)i(y) - i(x)i(r)i(y)$$
$$= \boxed{i(x)\Big(i(\ell) - i(r)\Big)i(y) \in P} \qquad (*)$$

Given the set of interpretations of letters $i(\Sigma) = A$, what is the set of admissible interpretations of rules $i(R) = B$?

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Given the set of interpretations of letters $i(\Sigma) = A$, what is the set of admissible interpretations of rules $i(R) = B$?
From $(*)$ it is obvious that $A^* B A^* \subseteq P$ is necessary.
The largest such set B is

$$\boxed{\mathrm{core}(A) = \{d \in D \mid A^* d A^* \subseteq P\}}$$

Example: For $A = \{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)\}$ we get $\mathrm{core}(A) = \{d \mid d \geq \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)\}$.

Core Facts

- Increasing the range of interpretations of letters typically reduces the set that can safely be chosen as interpretations of rules:

$$\boxed{\text{If } A_1 \subseteq A_2, \text{ then } \mathrm{core}(A_1) \supseteq \mathrm{core}(A_2)}$$

- The range of all interpretations is upward closed: W.l.o.g. for the interpretation of letters by

$$\boxed{\mathrm{core}(A + N) = \mathrm{core}(A)}$$

and for the interpretation of rules by

$$\boxed{\mathrm{core}(A) + N = \mathrm{core}(A)}$$

Termination via Interpretations (cont'd)

Let R be a string rewriting system over Σ.
An interpretation $i : \Sigma \to N$ into a p.o.-ring is order preserving

- from $(\Sigma^*, \to_R)$ to $(D, >)$ iff $i(R) \subseteq \mathrm{core}(i(\Sigma))$

Definition: Let A be a subset of the positive cone of a well-founded ring. Then $i : \Sigma \to A$ is an *A-interpretation* for R if

$$\boxed{i(R) \subseteq \mathrm{core}(A)}$$

Theorem:

- If there is an A-interpretation for R, then R is terminating.

Termination via Interpretations (cont'd)

Let R, S be string rewriting systems over Σ.
An interpretation $i : \Sigma \to N$ into a p.o.-ring is order preserving

- from $(\Sigma^*, \to_R)$ to $(D, >)$ iff $i(R) \subseteq \mathrm{core}(i(\Sigma))$
- from $(\Sigma^*, \to_S)$ to $(D, \geq)$ iff $i(S) \subseteq N$

Definition: Let A be a subset of the positive cone of a well-founded ring. Then $i : \Sigma \to A$ is an *A-interpretation* for R if

$$\boxed{i(R) \subseteq \mathrm{core}(A)}$$

Theorem:

- If there is an A-interpretation i for R with $i(S) \subseteq N$, then R is terminating relative to S.

Matrix Interpretations

Consider the p.o. ring of square matrices of a fixed dimension n over the integers: $\boxed{D = \mathbb{Z}^{n\times n}}$

- Addition / multiplication as usual.
- 0 and 1 are the zero and the identity matrix resp.
- The order is defined component-wise:
  $d \geq e$ if $\forall i, j : d_{i,j} \geq e_{i,j}$.
- The positive cone is $N = \mathbb{N}^{n\times n}$, and $P = N \setminus \{0\}$.
- The p.o. is well-founded on the positive cone.
- For $n > 1$, the p.o. is not total.

In order to apply the previous theorem we need a set of matrices $A \subseteq N$ with $\boxed{\text{non-empty } \mathrm{core}(A)}$.

Matrix Classes

Two particular instances of the above method:

- Choose $A = M_I$ with $\mathrm{core}(A) = M_I$.
- Choose $A = E_I$ with $\mathrm{core}(A) = P_I$.

All these are simple "syntactically" defined subsets of N, parameterized by a set of matrix indices $I \subseteq \{1, \ldots, n\}$:

$$\begin{aligned} M_I &= \{d \in N \mid \forall i \in I \exists j \in I : d_{i,j} > 0\} \\ E_I &= M_I \cap M_I^{\mathrm{T}} \\ P_I &= \{d \in N \mid \exists i \in I \exists j \in I : d_{i,j} > 0\} \end{aligned}$$

Consider only entries $d_{i,j}$ with $i, j \in I$:

- M_I: no zero row
- E_I: no zero row, no zero column

Example: $\{aa \to aba\}/\{b \to bb\}$

$$i(a) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad i(b) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is an E_1-interpretation with
$i(aa \to aba) = i(aa) - i(aba) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in P_1$
and $i(b \to bb) = i(b) - i(bb) = 0 \in N$.

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and $i(b \to bb) = i(b) - i(bb) = 0 \in N$.

Alternatively, use the M_2-interpretation

$$i(a) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad i(b) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

with $i(aa \to aba) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 2 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in M_2$ and
$i(b \to bb) = 0$. (This interpretation is not E_I for any I.)

Example: $\{aabb \to bbbaaa\}$

$$a \mapsto \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix} \qquad b \mapsto \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$$(\ell \to r) \mapsto \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is an $E_{\{1,5\}}$-interpretation.

Example: Linear Interpretations

- All termination proofs by $\boxed{\text{additive natural weights}}$ can be expressed as matrix interpretations:
  $(\mathbb{N}, +)$ is isomorphic to $(\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{N}\}, \cdot)$ since

  $$\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & m+n \\ 0 & 1 \end{smallmatrix}\right)$$

- More general: $\boxed{\text{Linear interpretations}}$
  - Interpret letters by functions $\lambda n.an + b$ on $\mathbb{N}$ with $a, b \in \mathbb{N}$ and $a \geq 1$,
  - concatenation is interpreted by function composition,
  - proof obligation is $\forall n : i(\ell)(n) > i(r)(n)$.

  This corresponds to matrix interpretations with matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$.

A Normal Form for E_I-Proofs

Matrix interpretations are invariant under permutations:

- If i is an E_I- or M_I-interpretation for R,
- and if π is a permutation on the index set $\{1, \ldots, n\}$,
- then there is also an $E_{\pi(I)}$- / $M_{\pi(I)}$-interpretation for R.

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$\Rightarrow$ W.l.o.g. we can replace an arbitrary set I by $\{1, \ldots, |I|\}$.

$\Rightarrow$ A normal form: Choose $J = \{1, n\}$.

- A proof of $\mathrm{SN}(R/S)$ via some E_I-interpretation can be replaced by a sequence of E_J-interpretations which successively remove the same rules.

Implementations: Performance

Percentage of YES in the 2006 SRS competition:

- MultumNonMulta (D.H.) 51 %
  matrix interpretations only
- Matchbox/Satelite (J.W.) 68 %
  labelling, matrices, RFC match-bounds
- TORPA (Hans Zantema) 75 %
  various techniques, including 3×3 matrices
- Jambox (Jörg Endrullis) 94 %
  $\approx$ Matchbox + dependency pairs

Implementations: TORPA

Random guesses or complete enumeration, using matrix shape

$$\begin{pmatrix} 0 & * & \boxed{+} \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix} \subseteq \mathrm{core} \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

with $* \in \{0, 1, 4\}$. Occurs in 36% of its proofs, e.g. z007:

```
TORPA 1.6 is applied to
a b  -> b a , b a  -> a a c b ,
[A] Choose interpretation in NxN,
order :  (x,y) > (x',y') <==> x > x' & y >= y'
a : lambda (x,y) . (x+y,4y)
b : lambda (x,y) . (x,4y+1)
c : lambda (x,y) . (x,0)
remove: a b  -> b a
```


Implementations: MultumNonMulta

- Random guesses, random restart hill climbing; complete enumeration, ... (not in the competition version)
- Backward completion, see below.
  - Examples: z061 / z062 / ...
  - Example: Waldmann/r10

    $$\mathrm{SN}(\{ba^2b \to a^4, ab^2a \to b^4\}/\{b \to b^3\})$$

    Sparse 14×14 matrices, found in 250 seconds.
- Determine additive weights using the GNU Linear Programming Kit.
  - Example: z112 / ...

Implementations: SAT Solving

- Fix dimension, say 5 $\rightsquigarrow$ $\boxed{\text{Constraint system}}$
  - $|\Sigma| \cdot d^2$ unknowns (matrix entries) and
  - $|R| \cdot d^2$ constraints (entries in differences).
- Fix maximal value for entries, say $7 = 2^3 - 1 \rightsquigarrow$ $\boxed{\text{Finite domain constraint system}}$
  - Binary encoding of entries $\rightsquigarrow$ boolean SAT problem: e.g. 15.000 variables, 90.000 clauses, 300.000 literals
  - Solve by SAT solver, e.g. SatELiteGTI.
    Expl: z001 takes 7 seconds
- Jambox: Linear programming + SAT solving.
- Matchbox: Likewise, but using only one bit per entry: Computation in $\{0, 1\} \subset \mathbb{N}$, so $1 + 1$ is “forbidden”.

Limitation: Derivational complexity

In a product of k matrices from a finite set,
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$\Rightarrow$ There can be no strict matrix interpretation for R.
Expl: $\{ab \to baa,\ cb \to bbc\}$

- Derivational complexity doubly exponential.
- But: “Relative” matrix proof with step-wise removal of rules is possible (first remove $cb \to bbc$).

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- Derivational complexity doubly exponential.
- But: "Relative" matrix proof with step-wise removal of rules is possible (first remove $cb \to bbc$).

$\Rightarrow$ There can be no matrix interpretation at all for R
if each rule occurs "equally often".
Expl: $\{ab \to bca, cb \to bbc\}$ (z018, z020)

- Derivational complexity tower of exponentials.
- But: Terminating via DP + matrix interpretations
- (and RPO-terminating).

Limitation: Dimension restrictions

A matrix ring is not *free*: Certain polynomial identities hold.

- Dimension 1: $[A, B] = 0$
  where $[A, B] = AB - BA$ (*commutator*)
  $\Rightarrow$ No 1-dim termination proof for $\{ab \to ba\}$.

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- Dimension 2: $[[A, B]^2, C] = 0$
  $\Rightarrow$ No 2-dim termination proof for
  $\{abcbc \to cbcba, acbcb \to bcbca, bccba \to abccb, cbbca \to acbbc\}$
  (Is RFC match-bounded. Matrix proof not known.)

Similar identities hold for matrix rings of any dimension.

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Similar identities hold for matrix rings of any dimension.

Define SRS hierarchy by "minimal matrix proof dimension":

- Is every level inhabited?
- Which levels are decidable?

Proof Verification

- Although probably hard to find, a termination proof via matrix interpretations is easy to verify ...
- ... and verification is fast: PTIME

Proof Verification

- Although probably hard to find, a termination proof via matrix interpretations is easy to verify ...
- ... and verification is fast: PTIME
- Even if the matrix type is not "syntactically" specified:
  - It is decidable whether an arbitrary matrix interpretation i satisfies $i(R) \subseteq \operatorname{core}(i(\Sigma))$.
  - Even more: we can effectively determine a finite set $C \subseteq P$ such that $\operatorname{core}(i(\Sigma)) = \{d \geq c \mid c \in C\}$.

Weighted Automata

Transitions have a natural number as *weight*:

A *weighted automaton* "is" a mapping $Q \times \Sigma \times Q \to \mathbb{N}$.

This mapping is extended to $Q \times \Sigma^* \times Q \to \mathbb{N}$:

- multiply weights along a single path,
- add weights of different paths.

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- multiply weights along a single path,
- add weights of different paths.

W.l.o.g. $Q = \{1, \ldots, n\}$.
For a transition from state p to state q with weight n
for letter a, the following representations are equivalent:

- State diagram: $p \xrightarrow{a:n} q$
- Matrix interpretation: $i(a)_{p,q} = n$

Weighted Automata (cont'd)

- Matrix multiplication computes the transitive closure:

  For $x \in \Sigma^*$, the $\boxed{\text{weight of path } (p) \xrightarrow{x} (q) \text{ is } i(x)_{p,q}}$

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  For $x \in \Sigma^*$, the $\boxed{\text{weight of path } (p) \xrightarrow{x} (q) \text{ is } i(x)_{p,q}}$

- "Standard" automata: $Q \times \Sigma \times Q \to \{0,1\}$.

- Other (semi-)rings possible ...

Zantema's System (cont'd)

The above matrix interpretation:

$$a \mapsto \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix} \qquad b \mapsto \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

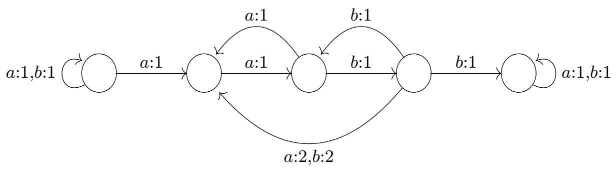
$$(\ell \to r) \mapsto \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

proves termination since

- *all* entries are ≥ 0 and
- *marked* entries are $\boxed{\geq 1}$

Zantema's System (cont'd)

The same termination proof as a weighted automaton:



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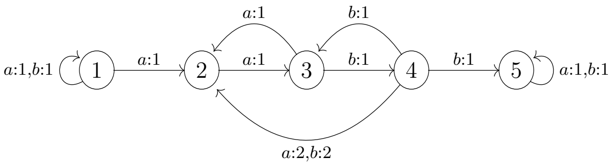
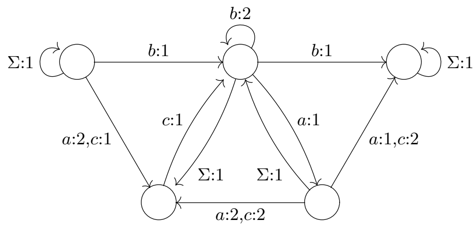


Example: $\{aa \to bc,\, bb \to ac,\, cc \to ab\}$

Solution for RTA List of Open Problems $\#104$:



A variant was published as a *monotone algebra* in IPL'06.

Automata: Large and Sparse

- Example: $\{bbcabc \to abbcbca\}$ (z061)

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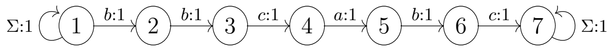
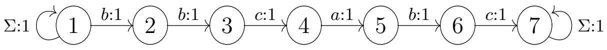


Done.

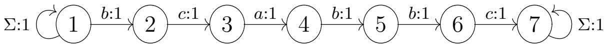
Automata: Large and Sparse

- Example: $\{bbcabc \to abbcbca\}$ (z061)



Done.

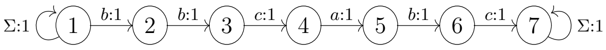
- Example: $\{bcabbc \to abcbbca\}$ (z062)



No: $\text{weight}\left(1 \xrightarrow{bcabbc} 4\right) = 0 \ngeq 1 = \text{weight}\left(1 \xrightarrow{abcbbca} 4\right)$

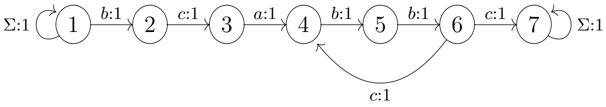
Automata: Large and Sparse

- Example: $\{bbcabc \to abbcbca\}$ (z061)



Done.

- Example: $\{bcabbc \to abcbbca\}$ (z062)



Done: $\text{weight}\left(1 \xrightarrow{bcabbc} 4\right) = 1 = \text{weight}\left(1 \xrightarrow{abcbbca} 4\right)$

Discussion

- Matrix interpretations for term rewriting:
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- Dependency pairs [Arts, Giesl 2000]:

$$\mathrm{SN}(R) \quad \text{iff} \quad \mathrm{SN}(\mathrm{DP}(R)/R)$$

  - The matrix method supports relative termination ⇒ it supports this basic version of the DP method.
  - Marker symbols encode the idea that $\mathrm{DP}(R)$ steps only happen at the left end (for terms: top position). [IJCAR 2006]: the matrix method can be adapted to relative top-termination
  - and can be combined with refinements [Hirokawa, Middeldorp 2004].

Future work

- Further instances of the general scheme are conceivable:
  Other matrix classes?
- Explain the relationship between proofs
  via E_I and via M_I.
- Explain the relationship between proofs
  via M_I and via $M_{I'}$ for $I \neq I'$.
- A normal form for M_I-proofs?
- Matrix interpretations are weighted finite automata.
  The method of (RFC) match-bounds also builds on weighted (annotated) automata.
  Unified view of these methods? (⇝ tomorrow)
- Good heuristics for backward completion.