Complexity bounds from relative termination proofs

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1-1

Derivational Complexity: Definition

The derivational complexity of a terminating (rewrite) relation $\to$ on a set of terms T is a mapping $\mathrm{dc} : \mathbb{N} \to \mathbb{N}$ with

$$\mathrm{dc}_\to : n \mapsto \max\{m \mid \exists x, y \in T : |x| \le n \wedge x \to^m y\}$$

where $|\_| : T \to \mathbb{N}$ is an appropriate size measure.

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- $\mathrm{dc}_\to(n) \in \Omega(n)$ for non-empty $\to$ (uses "context")

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Bridges to proof theory:

- Complexity bounds from termination proofs ...
- In general, $\mathrm{dc}_\to$ is an ordinal ...

String Rewriting: Definitions

- Letter: element of a set Σ, the alphabet
- String: sequence of letters. Σ^* is the set of strings over Σ
- String rewriting system: set of rules of the form $\ell \to r$, i.e. a set $R \subseteq \Sigma^* \times \Sigma^*$
- Rewrite step: replace the left hand side of rule $\ell \to r$ by its right hand side: $x\ell y \to_R xry$ within context $x, y \in \Sigma^*$
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Why study string rewriting in this context?

- A particular case of more general computation models: term / higher-order / graph / ... / rewriting systems

String Rewriting: Example

$R = \{aa \to bc,\ bb \to ac,\ cc \to ab\}$ induces e.g. the derivation

$$\begin{array}{l}
b\,b\,\boxed{a\,a} \to_R \\
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  No: Termination proof by a matrix interpretation.
  Yields exponential upper bound on dc_R.
- Open problem: polynomial upper bound?

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We can deduce some of these bounds automatically:

1. via match bounds
2. via upper triangular 3×3 matrix interpretations
3. via matrix interpretations

Research Program

Deduce upper bounds on the derivational complexity
from a termination proof (for general term rewriting).

- Multiset path order: primitive recursive [H]
- Lexicographic path order: multiple recursive [Weiermann]
- Knuth-Bendix order:
  4-recursive [H] / 2-recursive [Lepper]
- Related work by Buchholz, Touzet, Weiermann, Moser, ...
- Match bounds: linear [Geser, H, Waldmann]
- Matrix interpretations: exponential [H, Waldmann],
  polynomial in particular cases

Relative Termination

Let $S = \{ab \to baa\}$, $R = \{cb \to bbc\}$.
Consider R-steps in $R \cup S$-derivations.

The interpretation $\Sigma \to (\mathbb{N} \to \mathbb{N})$ with

$$a \mapsto \lambda n.n \qquad b \mapsto \lambda n.n+1 \qquad c \mapsto \lambda n.3n$$

is $\boxed{\text{constant}}$ for S and $\boxed{\text{decreasing}}$ for R
$\Rightarrow$ number of R-steps is $2^{O(n)}$.

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Relative termination allows to remove rules successively $\rightsquigarrow$

- Modular termination proofs
- Automatic methods for proving relative termination are incorporated in all state of the art termination provers.
- $\rightsquigarrow$ Annual termination competition [WST]

Relative Termination: Definition

System R is terminating relative to system S
if any $R \cup S$-derivation contains only finitely many R-steps.

- Notation: $\mathrm{SN}(R/S)$
- In other words: $\to_S^* \circ \to_R \circ \to_S^*$ is well-founded.

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Basic fact:

$$\boxed{\mathrm{SN}(R/S) \text{ and } \mathrm{SN}(S) \text{ imply } \mathrm{SN}(R \cup S)}$$

Expl: $\{aa \to aba\}$ is terminating relative to $\{b \to bb\}$.

The Problem

Let R and S be rewriting systems.
Assume termination of $R \cup S$ has been shown
by proving termination of R/S and termination of S.

- $\boxed{\text{Give a bound on } \mathrm{dc}_{R \cup S} \text{ in terms of } \mathrm{dc}_{R/S} \text{ and } \mathrm{dc}_S.}$

Note: Proof methods for relative termination
can handle situations where S is not terminating.
Here we assume that S *is* terminating.

Basic Observation

Let $\Delta_R = \max\{|r| \dot{-} |\ell| \mid (\ell \to r) \in R\}$, and assume (for simplicity) that this implies $\max\{|x| \dot{-} |y| \mid x \to_R y\} \le \Delta_R$.

- Note: $\Delta_R = 0$ in case R is not size-increasing.

Now consider an arbitrary finite derivation modulo $R \cup S$:

$$\boxed{x_0 \to_S^* x_0' \to_R x_1 \to_S^* x_1' \to_R x_2 \to_S^* \cdots \to_S^* x_{k-1}' \to_R x_k \to_S^* x_k'}$$

Define $\delta : \mathbb{N} \to \mathbb{N}$ by $\delta(n) = n + \Delta_S \cdot \mathrm{dc}_S(n) + \Delta_R$. Then

$$|x_{i+1}| \le \delta(|x_i|).$$

Monotonicity of dc_S implies monotonicity of δ, thus

$$|x_{i+1}| \le \delta^i(|x_0|).$$

The General Upper Bound

$$\boxed{x_0 \to_S^* x_0' \to_R x_1 \to_S^* x_1' \to_R x_2 \to_S^* \cdots \to_S^* x_{k-1}' \to_R x_k \to_S^* x_k'}$$

...thus the length of the above derivation is bounded by

$$\mathrm{dc}_{R \cup S}(|x_0|) \le \mathrm{dc}_{R/S}(|x_0|) + \sum_{i=0}^{k} \mathrm{dc}_S(|x_i|)$$
$$\le \mathrm{dc}_{R/S}(|x_0|) + \sum_{i=0}^{k} \mathrm{dc}_S\left(\delta^i(|x_0|)\right)$$

We have $\delta^{i+1}(n) \ge \delta^i(n)$ by $\delta(n) \ge n$. Since $k \le \mathrm{dc}_{R/S}(|x_0|)$,

$$\boxed{\mathrm{dc}_{R \cup S}(n) \in O\Big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S\left(\delta^{\mathrm{dc}_{R/S}(n)}(n)\right)\Big)}$$

Particular Cases

- R and S not size-increasing: $\delta(n) = n$
  $\boxed{\mathrm{dc}_{R \cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S(n)\big)}$
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- S not size-increasing:
  $\delta(n) = n + \Delta_R$, thus $\delta^i(n) = n + i \cdot \Delta_R$
  $\boxed{\mathrm{dc}_{R \cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S(n + \mathrm{dc}_{R/S}(n) \cdot \Delta_R)\big)}$
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- S size-increasing: $\delta \in \Theta(\mathrm{dc}_S)$

  $\boxed{\mathrm{dc}_{R\cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S^{\mathrm{dc}_{R/S}(n)+1}(n)\big)}$

  Iteration

Consequences

- Consider function classes with certain closure properties:
  - Closed under addition, multiplication, composition
    Example: polynomials
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    Example: polynomials
  - Closed under iteration
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- Can this general bound be improved?
  No, as the following generic construction reveals.
  (For string rewriting, therefore can be done in every sufficiently rich rewriting model.)

The Lower Bound Result

The general upper bound can be attained,
even for string rewriting. Proof:

Take arbitrary string rewriting systems R_0 over Σ, S_0 over Γ (w.l.o.g. disjoint alphabets) and add new letters σ, γ. Define

$$R = \{l \to r\sigma \mid (l \to r) \in R_0\} \qquad \text{(introduce marker)}$$
$$S = S_0 \cup \{\sigma a \to a\sigma \mid a \in \Sigma\} \qquad \text{(move marker)}$$
$$\cup\, \{\sigma \to \gamma\} \qquad \text{(switch markers)}$$
$$\cup\, \{\gamma b \to c\gamma \mid b, c \in \Gamma\} \qquad \text{(nondeterministic reset)}$$

We have $\boxed{\mathrm{dc}_{R_0} \approx \mathrm{dc}_{R/S}}$, $\boxed{\mathrm{dc}_{S_0} + \Theta(n^2) \approx \mathrm{dc}_S}$ and $\boxed{\mathrm{dc}_{R\cup S} = \Theta(\text{upper bound in terms of } \mathrm{dc}_{R/S} \text{ and } \mathrm{dc}_S)}$.

So the construction shows optimality if $\mathrm{dc}_S \in \Omega(n^2)$.

Example: Polynomial Upper Bound

$$B_k = \{ki \to jk \mid k > i, j\}$$
$$R_d = B_2 \cup \cdots \cup B_d$$

over alphabet $\{1, 2, \ldots, d\}$. The bound $\boxed{\mathrm{dc}_{R_d} \in \Theta(n^d)}$ can be shown via some matrix interpretation of dimension $d + 1$.

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A simpler proof via relative termination:

- Show $\mathrm{SN}(B_d/R_{d-1})$ via the interpretation
  $\{1, \ldots, d-1\} \mapsto \begin{pmatrix} 1&0&0 \\ 0&1&1 \\ 0&0&1 \end{pmatrix}, \qquad d \mapsto \begin{pmatrix} 1&1&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$
- $\mathrm{dc}_{B_d/R_{d-1}} \in O(n^2)$ (matrices are upper triangular)
- B_d and R_{d-1} are size-preserving, so the upper bound result implies (by induction) $\boxed{\mathrm{dc}_{R_d} \in O(n^{2(d-1)})}$.

Bound is overestimated, but nevertheless polynomial.
Termination proof much easier to find.

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 Yes for term rewriting:

$$R = \{f(s(x), y, z) \to f(x, z, y) \mid x, y, z \geq 0\}$$
$$S = \{f(x, s(y), z) \to f(x, y, s(s(z))) \mid x, y, z \geq 0\}$$

Here, $\mathrm{dc}_{R/S} \in O(n)$ and $\mathrm{dc}_S \in O(n)$, but $\mathrm{dc}_{R\cup S}$ is exponential: $f(s^n(0), 1, 0) \to^* f(0, 0, s^{2^n}(0))$.

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- Make the implicit notion of "abstract reduction system with size measure" explicit.

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Reminder (after the talk)

Solving the above question:
There are also string rewriting systems
with $\mathrm{dc}_{R/S} \in O(n)$ and $\mathrm{dc}_S \in O(n)$, but $\mathrm{dc}_{R \cup S}$ exponential:

$$R = \{c\lhd \to \rhd\}$$
$$S = \{\rhd a \to bb\rhd,$$
$$\rhd \to \lhd,$$
$$b\lhd \to \lhd a\}$$

We have

$$c^n \rhd a \to^*_{R \cup S} \rhd a^{2^n}$$

(Note that S is match-bounded by 2.)

17-1

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Rewriting and Proof Theory, Obergurgl, 6. September 2006 – p.18/30

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Here we assume that S *is* terminating.

Basic Observation

Let $\Delta_R = \max\{|r| \dot{-} |\ell| \mid (\ell \to r) \in R\}$, and assume (for simplicity) that this implies $\max\{|x| \dot{-} |y| \mid x \to_R y\} \leq \Delta_R$.

- Note: $\Delta_R = 0$ in case R is not size-increasing.

Now consider an arbitrary finite derivation modulo $R \cup S$:

$$\boxed{x_0 \to_S^* x_0' \to_R x_1 \to_S^* x_1' \to_R x_2 \to_S^* \cdots \to_S^* x_{k-1}' \to_R x_k \to_S^* x_k'}$$

Define $\delta : \mathbb{N} \to \mathbb{N}$ by $\delta(n) = n + \Delta_S \cdot \mathrm{dc}_S(n) + \Delta_R$. Then

$$|x_{i+1}| \leq \delta(|x_i|).$$

Monotonicity of dc_S implies monotonicity of δ, thus

$$|x_{i+1}| \leq \delta^i(|x_0|).$$

The General Upper Bound

$$\boxed{x_0 \to_S^* x_0' \to_R x_1 \to_S^* x_1' \to_R x_2 \to_S^* \cdots \to_S^* x_{k-1}' \to_R x_k \to_S^* x_k'}$$

...thus the length of the above derivation is bounded by

$$\begin{aligned} \mathrm{dc}_{R\cup S}(|x_0|) &\leq \mathrm{dc}_{R/S}(|x_0|) + \sum_{i=0}^{k} \mathrm{dc}_S(|x_i|) \\ &\leq \mathrm{dc}_{R/S}(|x_0|) + \sum_{i=0}^{k} \mathrm{dc}_S\left(\delta^i(|x_0|)\right) \end{aligned}$$

We have $\delta^{i+1}(n) \geq \delta^i(n)$ by $\delta(n) \geq n$. Since $k \leq \mathrm{dc}_{R/S}(|x_0|)$,

$$\boxed{\mathrm{dc}_{R\cup S}(n) \in O\Big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S\left(\delta^{\mathrm{dc}_{R/S}(n)}(n)\right)\Big)}$$

Particular Cases

- R and S not size-increasing: $\delta(n) = n$
  $\boxed{\mathrm{dc}_{R\cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S(n)\big)}$
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  Multiplication
- S not size-increasing:
  $\delta(n) = n + \Delta_R$, thus $\delta^i(n) = n + i \cdot \Delta_R$
  $\boxed{\mathrm{dc}_{R\cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S\left(n + \mathrm{dc}_{R/S}(n) \cdot \Delta_R\right)\big)}$
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  Composition

- S size-increasing: $\delta \in \Theta(\mathrm{dc}_S)$

  $\boxed{\mathrm{dc}_{R\cup S}(n) \in O\big(\mathrm{dc}_{R/S}(n) \cdot \mathrm{dc}_S^{\mathrm{dc}_{R/S}(n)+1}(n)\big)}$

  Iteration

Consequences

- Consider function classes with certain closure properties:
  - Closed under addition, multiplication, composition
    Example: polynomials
  - Closed under iteration
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Consequences

- Consider function classes with certain closure properties:
  - Closed under addition, multiplication, composition
    Example: polynomials
  - Closed under iteration
    Example: primitive recursive functions
- Can this general bound be improved?
  No, as the following generic construction reveals.
  (For string rewriting, therefore can be done in every sufficiently rich rewriting model.)

The Lower Bound Result

The general upper bound can be attained,
even for string rewriting. Proof:

Take arbitrary string rewriting systems R_0 over Σ, S_0 over Γ (w.l.o.g. disjoint alphabets) and add new letters σ, γ. Define

$$\begin{aligned} R &= \{l \to r\sigma \mid (l \to r) \in R_0\} && \text{(introduce marker)} \\ S &= S_0 \cup \{\sigma a \to a\sigma \mid a \in \Sigma\} && \text{(move marker)} \\ &\quad \cup \{\sigma \to \gamma\} && \text{(switch markers)} \\ &\quad \cup \{\gamma b \to c\gamma \mid b, c \in \Gamma\} && \text{(nondeterministic reset)} \end{aligned}$$

We have $\boxed{\mathrm{dc}_{R_0} \approx \mathrm{dc}_{R/S}}$, $\boxed{\mathrm{dc}_{S_0} + \Theta(n^2) \approx \mathrm{dc}_S}$ and

$\boxed{\mathrm{dc}_{R\cup S} = \Theta(\text{upper bound in terms of } \mathrm{dc}_{R/S} \text{ and } \mathrm{dc}_S).}$

So the construction shows optimality if $\mathrm{dc}_S \in \Omega(n^2)$.

Example: Polynomial Upper Bound

$$B_k = \{ki \to jk \mid k > i, j\}$$
$$R_d = B_2 \cup \cdots \cup B_d$$

over alphabet $\{1, 2, \ldots, d\}$. The bound $\boxed{\mathrm{dc}_{R_d} \in \Theta(n^d)}$ can be shown via some matrix interpretation of dimension $d + 1$.

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A simpler proof via relative termination:

- Show $\mathrm{SN}(B_d/R_{d-1})$ via the interpretation
  $\{1, \ldots, d-1\} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\mathrm{dc}_{B_d/R_{d-1}} \in O(n^2)$ (matrices are upper triangular)
- B_d and R_{d-1} are size-preserving, so the upper bound result implies (by induction) $\boxed{\mathrm{dc}_{R_d} \in O(n^{2(d-1)})}$.

Bound is overestimated, but nevertheless polynomial.
Termination proof much easier to find.

> 1.3