“Free” SCC Analysis via Constant Interpretations

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Motivation

- implementation of dependency pairs method
- that constructs (something like) the DP graph and its strongly connected components
- from (matrix) interpretations (found via SAT solver)
- with very little additional implementation cost

(this is the method of Matchbox/TRS in 2006)

DP Method

. . . transforms a standard termination problem

into a *relative top*-termination problem:

$\mathrm{SN}(\to_R)$ is equivalent to $\mathrm{SN}(\mathrm{DP}(R)_{\mathrm{top}}/R)$.

Example: $R = \{aa \to aba\}$ over $\Sigma = \{a, b\}$,

then $\mathrm{DP}(R) = \{Aa \to Aba, Aa \to A\}$

over $\Sigma \cup \Sigma'$ with $\Sigma' = \{A, B\}$.

Interpretations for DP Problems

alphabets Σ (original) and Σ' (defined symbols)
two-sorted algebra with sorts $(S, \gtrsim)$ and $(T, >)$
interpretation $[\cdot]$: $\Sigma \to (S^* \to S), \Sigma' \to (S^* \to T)$

- each $[f]$ weakly monotone in each argument w.r.t. $\gtrsim$ resp. $\geq$
- $\forall (l \to r) \in R : \forall \alpha \in \mathrm{Var} \to A : [l, \alpha] \gtrsim [r, \alpha]$
- $\forall (l \to r) \in D : \forall \alpha \in \mathrm{Var} \to A : [l, \alpha] > [r, \alpha]$,

implies $\mathrm{SN}(D_{\mathrm{top}}/R)$.

Matrix Interpretations for DP

sort $S =$ column vectors $\mathbb{N}^{1\times d}$, $T =$ naturals $\mathbb{N}^{1\times 1}$.

order $\gtrsim$ on S component-wise, $>$ on T standard.

interpretation $[f]$ is linear function

$$[f](x_1, \ldots, x_k) = M_1 \cdot x_1 + \ldots + M_k \cdot x_k + v.$$

for matrices $M_1, \ldots, M_k \in \mathbb{N}^{e\times d}$, vector $v \in \mathbb{N}^{e\times 1}$, for $e \in \{d, 1\}$.

interpretations $[l, \alpha], [r, \alpha]$ are also linear functions

weak monotonicity: $\geq$ for pairs of coefficients,
strict monotonicity: $>$ in absolute part

Splitting DP Problems

consider such an interpretation where

$\forall(l \longrightarrow r) \in D$, $[l, \alpha]$ and $[r, \alpha]$ are *constant*

($=$ do not depend on value of variables α)

level h of D, written D_h,

consists of all rules $(l \rightarrow r) \in D$
where $[l, \alpha] = [r, \alpha] = \mathrm{const}\, h$.

$$\mathrm{SN}(D_{0,\mathrm{top}}/R) \wedge \ldots \wedge \mathrm{SN}(D_{k,\mathrm{top}}/R) \iff \mathrm{SN}(D_{\mathrm{top}}/R)$$

Example (I)

$$R = \{ab \to a^3, b^3 \to a^2ba^2, bab^2 \to b^3ab\}.$$

$$D = \left\{ \begin{array}{c} Ab \to Aa^{0,1,2}, \\ Bb^2 \to Aa^{0,1}ba^2, Bb^2 \to Ba^2, Bb^2 \to Aa^{0,1}, \\ Bab^2 \to Bb^{0,1,2}ab, Bab^2 \to Ab, \end{array} \right.$$

0-dimensional interpretation (vectors of length 0 for sort S) and $[A](x) = 0, [B](x) = 1$:

- level one: $\{Bb^2 \to Ba^2, Bab^2 \to Bb^{0,1,2}ab\}$,
- level zero: $Ab \to Aa^{0,1,2}$

ignore decreasing rules $B \ldots \to A \ldots$

Example (II)

For level zero ($Ab \to Aa^{0,1,2}$)
use interpretation

$$
\begin{aligned}
a : x &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot x, \\
b : x &\mapsto \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \cdot x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
A : x &\mapsto \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot x.
\end{aligned}
$$

weakly monotonic for R, strictly monotonic for level zero of D

Example (III)

For level one $\{Bb^2 \to Ba^2, Bab^2 \to Bb^{0,1,2}ab\}$, use

$$a : x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot x + \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$b : x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B : x \mapsto \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot x.$$

weakly monotonic for $R \cup D$ *and* constant for D:

$\forall t \in \{Bb^2, Ba^2, Bb^{1,2}ab\} : [t](x) = 0$

$\forall t \in \{Bab^2, Bab\} : [t](x) = 1.$

Remove (decreasing) $\{Bab^2 \to Bb^{1,2}ab\}$ and split :

$$\mathrm{SN}(Bb^2 \to Ba^3/R) \quad \text{and} \quad \mathrm{SN}(Bab^2 \to Bab/R).$$

Discussion (Example)

Termination of R cannot be shown by “pure” dependency pair approach (Aprove, TTT give up)

There is a termination proof via labelling w.r.t. a (quasi) model in $\{0, 1\}^2$ (found by Torpa-1.4 and TPA-1.0)

and there is a 4×4-matrix interpretation (found by the Xbox provers).

Splitting via constant interpretations helps to reduce the proof obligations, as the matrix dimension is reduced from 4×4 to 2×2.

Discussion (general)

- method can (to some extent) replace SCC analysis of DP graph
- implementation is trivial for provers that already have a constraint solver that finds (matrix) interpretations.
- method is “verifier-friendly”

The exact relation between our splitting construction and standard algorithms remains open.

Literatur

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