Equivalence of Match-Boundedness, Change-Boundedness and inverse Match-Boundedness for Length-Preserving String Rewriting

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String Rewriting

string rewriting system R
over alphabet Σ is set of rules,
rule is pair of words. $R \subseteq \Sigma^* \times \Sigma^*$.

$R = \{abb \to bba, bbb \to aaa\}$
over $\Sigma = \{a, b\}$.

rewrite relation: apply rule in context. $u \to_R v \iff$
$\exists x, y \in \Sigma^*, (l \to r) \in R :$
$u = xly \wedge xry = v$

derivation $u_0 \to_R u_1 \to_R \dots$

example derivation

```
abbbb
---
bbabb
  ---
bbbba
 ---
baaaa
```

Height Annotations (I)

change heights:
for each position p count number of rule applications that contain p

Bala Ravikumar: Peg-solitaire, string rewriting systems and finite automata, TCS 2004

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b2 b1 a1
   -- -- --
b1 a2 a3 a2 a1
```

bounded change heights imply:

- termination
- effective preservation of regularity

Height Annotations (II)

match heights:
for each rule application:
each height in reduct
is $1+$ minimal height in redex
Geser, Hofbauer, Waldmann: Match-Bounded String Rewriting Systems, AAECC 2004

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b1 b1 a1
   -- -- --
b1 a2 a2 a2 a1
```

bounded match heights imply:

- termination
- effective preservation of regularity

Height Annotations (III)

inverse match heights: match-heights for the inverse derivation

Geser, Hofbauer, Waldmann: Termination Proofs for String Rewriting Systems via Inverse Match-Bounds, JAR 2005.

```
a1 b1 b1 b1 b1
-- -- --
b0 b1 a1 b1 b1
      -- -- --
b0 b1 b1 b1 a0
   -- -- --
b0 a0 a0 a0 a0
```

bounded inverse match heights imply:

- termination is decidable
- effective preservation of CF

Central Question

relation between (existence and then value) of bounds for match, change, inverse match heights?

match

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b1 b1 a1
   -- -- --
b1 a2 a2 a2 a1
```

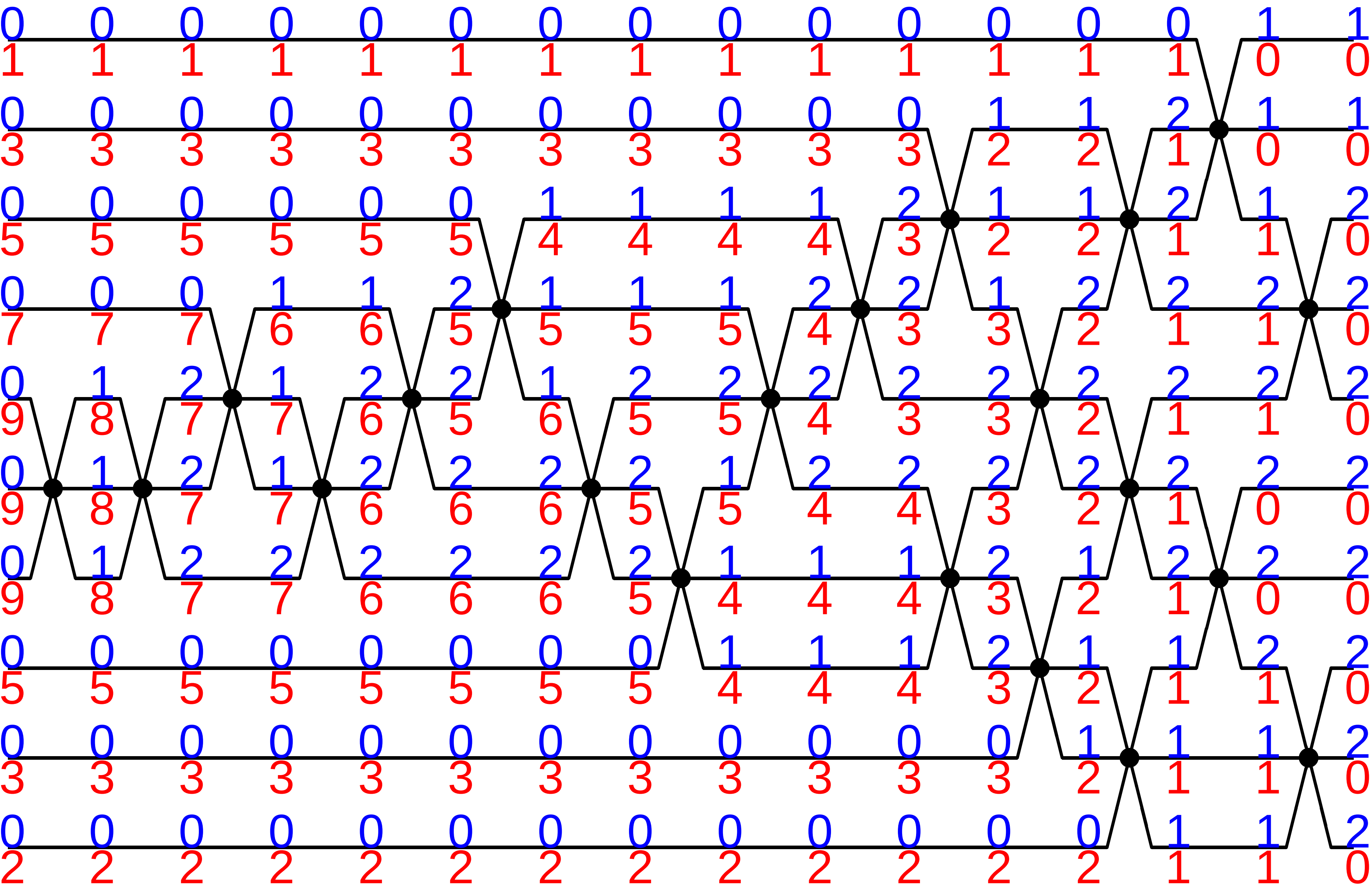
change

```
a0 b0 b0 b0 b0
-- -- --
b1 b1 a1 b0 b0
      -- -- --
b1 b1 b2 b1 a1
   -- -- --
b1 a2 a3 a2 a1
```

inverse match

```
a1 b1 b1 b1 b1
-- -- --
b0 b1 a1 b1 b1
      -- -- --
b0 b1 b1 b1 a0
   -- -- --
b0 a0 a0 a0 a0
```

Example ($w = 3$, $M = 2$, $M' = 9$)



Answer

for length-preserving systems R,
the following are equivalent:

1. R is match-bounded
2. R is change-bounded
3. R is inverse change-bounded
4. R is inverse match-bounded

Note: only need to show $1 \Rightarrow 2$, since $1 \Leftarrow 2$ is easy, and $2 \Rightarrow 3$ is trivial, rest follows by symmetry.
NB: This will *not* give a sharp bound for $1 \Rightarrow 4$.

Proof Sketch ($1 \Rightarrow 2$)

$$
\begin{array}{ccccccccc}
u_0 & \xrightarrow{R} & u_1 & \xrightarrow{R} & u_2 & \xrightarrow{R} & u_3 & \xrightarrow{R} & \\
\downarrow {\scriptstyle \mathrm{loc}_p} & & \downarrow {\scriptstyle \mathrm{loc}_p} & & \downarrow {\scriptstyle \mathrm{loc}_p} & & \downarrow {\scriptstyle \mathrm{loc}_p} & & \\
v_0 & \xrightarrow{S^{0,1}} & v_1 & \xrightarrow{S^{0,1}} & v_2 & \xrightarrow{S^{0,1}} & v_3 & \xrightarrow{S^{0,1}} &
\end{array}
$$

fix a position p, use *localization* (down arrows):

- if R step touches p, then local step S^1 (else $S^{0,1}$).
- $|v_0|$ is bounded (indep. of $|u_0|$)
- if top row is match-bounded, then bottom is mb.

$\Rightarrow$ number of S steps is bounded,

$\Rightarrow$ number of R steps that touch p is bounded,

$\Rightarrow$ R is change-bounded at p.

Tool: Min/+ Algebra

the *tropical* semi-ring on $\mathbb{N} \cup \{\infty\}$:

- addition $\oplus$ is $\min$, zero: ∞
- multiplication $\otimes$ is $+$, unit: 0

action of rewrite step on match heights
is a min/+ linear operator (matrix):

$$(1, 0, \underline{6, 2}, 5) \odot \begin{pmatrix} 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty \\ \infty & \infty & 1 & 1 & \infty \\ \infty & \infty & 1 & 1 & \infty \\ \infty & \infty & \infty & \infty & 0 \end{pmatrix} = (1, 0, \underline{3, 3}, 5)$$

cf: Tetris is max/+ linear, Gaubert et al, STACS 97

Localization (I)

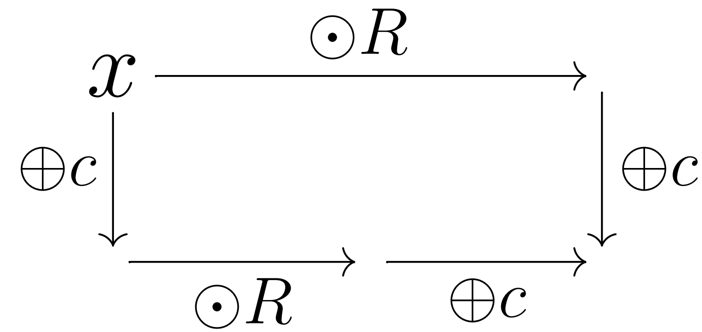
$\mathrm{loc}_p(x) := x \oplus c$ point-wise

for $c = 1^{w-1}2^{w-1}\dots M^{2w-1}\dots 2^{w-1}1^{w-1}$

centered at position p and cutting off

for any step: $c \odot R \geq c$, thus $c \odot R \oplus c = c$.

This diagram commutes for any x:



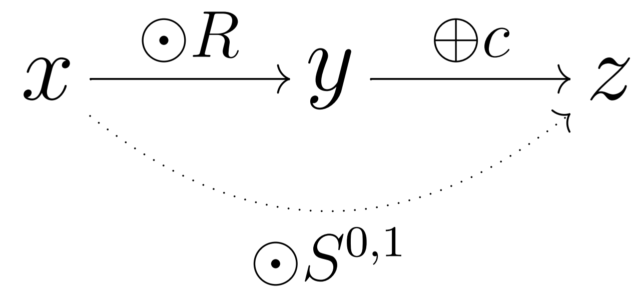
$$x \odot R \oplus c = x \odot R \oplus c \odot R \oplus c = (x \oplus c) \odot R \oplus c.$$

NB: if $p \in R$ (rewrite step that touches p),

then $x \neq c$. (uses match-boundedness of R-steps)

Localization (II)

For any $x \leq c$ and rewrite step R, there is some (possibly empty) rewrite step S such that:

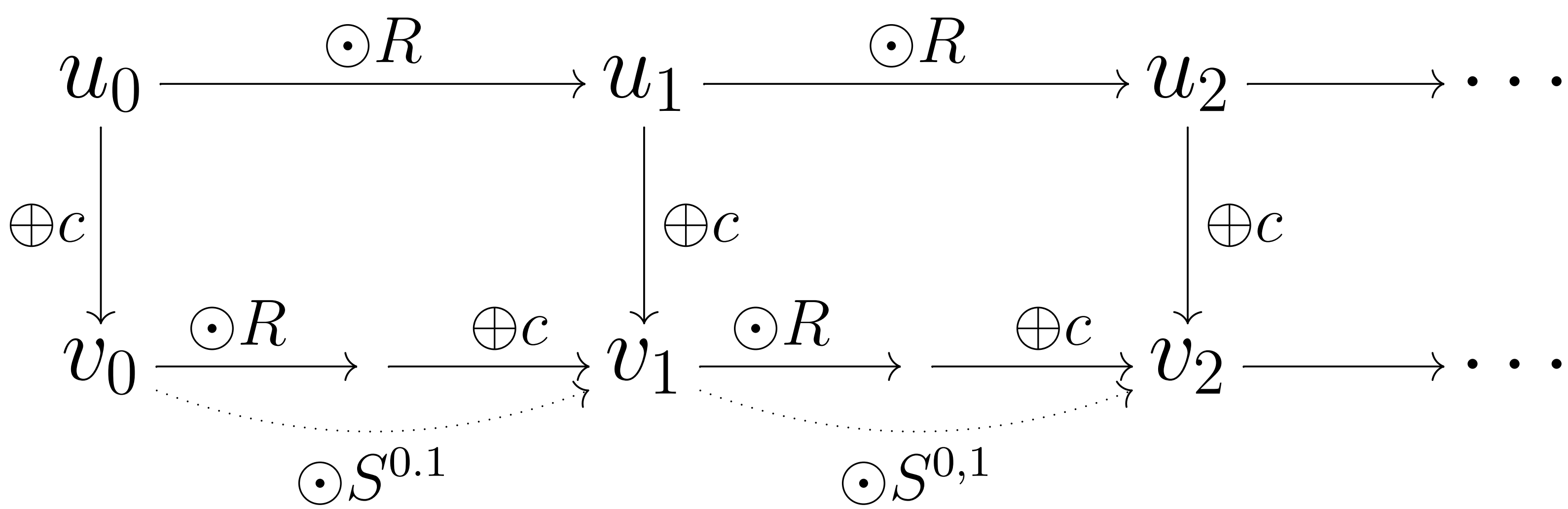


Example:
$$\begin{aligned}
w = 3, c &= (\ldots, 2, 3, 3, 4, 4, \ldots),\\
x &= (\ldots, 0, 0, \underline{3, 3, 4}, \ldots),\\
y = x \odot R &= (\ldots, 0, 0, 4, 4, 4, \ldots),\\
z = y \oplus c &= (\ldots, 0, 0, 3, 4, 4, \ldots).
\end{aligned}$$

The S-Redex in x is $(\ldots, 0, 0, 3, \underline{3, 4}, \ldots)$.

Proof: if $x =_{\operatorname{dom} R} c$, then S empty,
else $\operatorname{dom} S = \{q \in \operatorname{dom} R : y(q) \leq c(q)\}$.

Localization (III)



- R-derivation is match-bounded

$\Rightarrow$ S-derivation is match-bounded

- $|v_0|$ is fixed

$\Rightarrow$ length of S-derivation is bounded

$\Rightarrow$ R-derivation is change-bounded

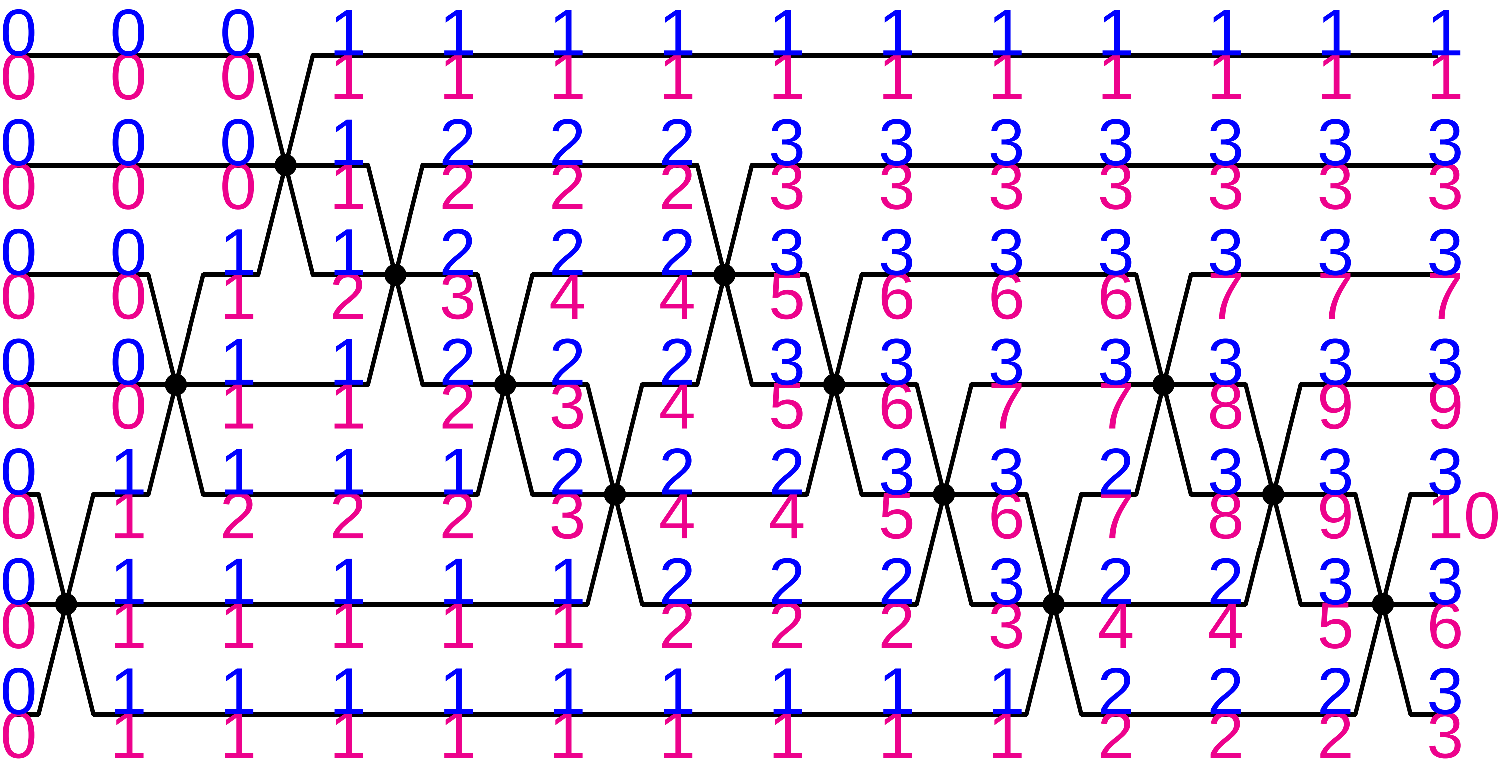
Bounds from the Proof

match-bounded (S) derivation on string of length l has $\leq l(w+1)^M$ steps

- original system R is change-bounded by $2wM(w+1)^M$.
  (this bound seems attainable)
- this also bounds inverse match-heights
  (but this bound seems to be far off)

on next slides: families of derivations with largest known growths

Large Change Heights (I)



$$D_w(0) \;=\; []$$

$$D_w(M+1) \;=\; [0]\cdot \uparrow_M (D_w(M)) \cdot \ldots \cdot \uparrow_0 (D_w(M))$$

e.g. $D_3(3) = [0,\ 2,4,3,2,\ 1,3,2,1,\ 0,2,1,0]$

Large Change Heights (II)

$$\begin{aligned} D_w(0) &= [] \\ D_w(M+1) &= [0] \cdot \uparrow_M (D_w(M)) \cdot \ldots \cdot \uparrow_0 (D_w(M)) \end{aligned}$$

- match-bounded by M
- length of derivation: $|D_w(M)| = \sum_{i=0}^{M} w^i$
- width of start string: $w \cdot M$
- by pigeonhole principle, there is position with change height $\geq w^M/(wM)$.
- cf. with upper bound from proof: $2wM(w+1)^M$

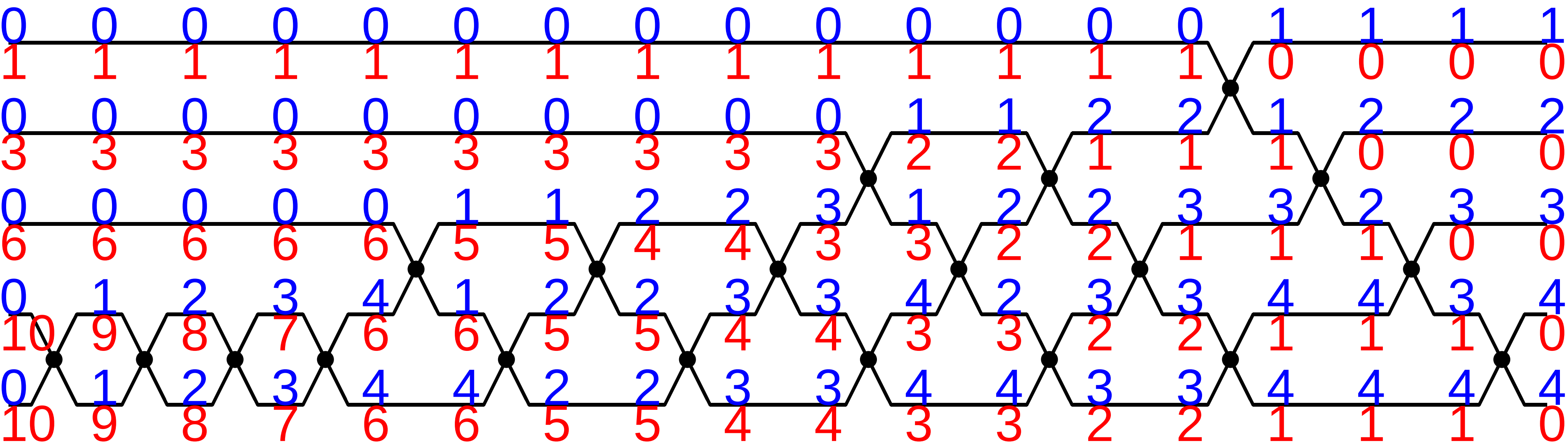
Large Inverse Match Heights (I)

given width of rule w , match-bound M
maximize inverse match-bound M'.

M'	$w = 1$	2	3	4	5
$M = 1$	1	2	2	2	2
2	2	6	9	12	15
3	3	12	20		
4	4	19			
5	5	26			

found by computer search

Large Inverse Match Heights (II)



$$E(2, M) = B(1, M) \cdot B(2, M-1) \cdot \ldots \cdot B(M, 1)$$

$$B(v, h) = [v, v-1, \ldots, 1]^h$$

match-bound: M, inv. mb: $M + \ldots + 1 = \Theta(M^2)$.

in general, inverse match-bound is $\Theta(wM^2)$.

Conclusion, Discussion

Conclusion

- min/plus algebra is a useful tool
  in match-bounded rewriting

Open Questions

- exact upper bounds for inverse match heights
  (polynomial wM^2 or exponential w^M ?)
- drop the length-preserving restriction
  (but then cannot use proof via change heights)