Arctic Termination

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Standard Matrix Method: Example

$$a \mapsto \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix} \qquad b \mapsto \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

$$(a^2b^2 \to b^3a^3) \mapsto$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This works because . . .

- $(\mathbb{N}, +, \cdot)$ is a semi-ring (associativity, distributivity) $\Rightarrow$ matrices form semi-ring
- $(\mathbb{N}, +, \cdot, <)$ is an *ordered* semi-ring (plus and times are weakly monotonic in each argument)
- $(\mathbb{N}, <)$ is well-founded
- plus is strict:
  $\forall x < y, z : x + z < y + z,$
- times is strict except at 0:
  $\forall x < y, z \neq 0 : x \cdot z < y \cdot z$

The Arctic Semi-Ring: Definition

a.k.a. the (max,plus) algebra

- domain: $\mathbb{A} := \{-\infty\} \cup \mathbb{N}$,
- operations:
  - Addition $x \oplus y = \max(x, y)$, neutral: $-\infty$
  - Multiplication $x \otimes y = x + y$, neutral: 0.

Trivia: why the name?

- (min,plus) on $\mathbb{N} \cup \{+\infty\}$ is the *tropical* semi-ring, in honour of inventor Imre Simon, living in Brazil
- (max,plus) is the opposite of (min,plus)

The Arctic Semi-Ring: Properties

- domain $\mathbb{A}$ is well-founded: $-\infty < 0 < 1 < \dots$
- multiplication (plus) is strict except at $-\infty$,
- addition (max) is *not strict* $3 < 4, 3 \oplus 5 \not< 4 \oplus 5$
- . . . but $2 < 3, 4 < 5, 2 \oplus 4 < 3 \oplus 5$

Operation $\oplus$ is called *half-strict*:

$$\forall a < b, c < d : a \oplus c < b \oplus d.$$

Arctic Matrices for Termination

- interpret letter $c \in \Sigma$ by matrix $[c]$ from domain $\{M : M \in \mathbb{A}^{d \times d}, -\infty < M_{1,1}\}$.
- order domain by $P > Q$ iff
  $\forall i, j : P_{i,j} > Q_{i,j} \vee P_{i,j} = -\infty = Q_{i,j}$.
- let $P \geq Q$ denote $\forall i, j : P_{i,j} \geq Q_{i,j}$.

Theorem:

If for each $(l \to r) \in R : [l] > [r]$,
and for each $(l \to r) \in S : [l] \geq [r]$,
then R is terminating relative to S.

An Arctic Termination Proof

$$[a] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} 0 & -\infty \\ -\infty & -\infty \end{pmatrix}$$

$$[a^2] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \max(0+0, 0+1) & \max(0+0, 0+1) \\ \max(1+0, 1+1) & \max(1+0, 1+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$> \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = [aba]$$

this proves $\mathrm{SN}(a^2 \to aba)$
. . . and $\mathrm{SN}(R/\{b \to b^2\})$ since $[b^2] = [b]$.

Arctic Matrices and DP Method

use $\mathbb{A}^{1 \times d}$ (row vectors) for top symbols.
z086: $R = \{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$
essentially $\mathrm{DP}(R)$ is $\{Aa \to Bc, Bb \to Ac\}$.

$$[a] = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, [b] = \begin{pmatrix} 3 & 2 \\ 1 & -\infty \end{pmatrix}, [c] = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, [A] = [B] = \begin{pmatrix} 0 & -\infty \end{pmatrix}$$

$$[Bb] = (3\ 2) > (0\ 1) = [Ac] \qquad \Leftarrow \text{remove this rule}$$

$$[Aa] = (0\ 3) \geq (0\ 1) = [Bc]$$

$$[a^2] = \begin{pmatrix} 5 & 4 \\ 3 & 5 \end{pmatrix} \geq \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} = [bc], \ [b^2] = \begin{pmatrix} 6 & 5 \\ 4 & 3 \end{pmatrix} = [ac]$$

$$[c^2] = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \geq \begin{pmatrix} 4 & 2 \\ 5 & 4 \end{pmatrix} = [ab]$$

Quasi-periodic interpretations

$R = \{b^3 \to a^3, a^3 \to bab\}$,
$\mathrm{DP}(R)$ reduced to $\{Bb^2 \to Aa^2, Aa^2 \to Bab\}$

QPI (period 3) maps $0, 1, 2, 3, 4, 5, \ldots$ to:

a	$2, 3, 4, 5, 6, 7, \ldots$	b	$3, 3, 3, 6, 6, 6, \ldots$
A	$2, 4, 5, 5, 7, 8, \ldots$	B	$3, 4, 4, 6, 7, 7, \ldots$

interpretation of rules

b^3	$9, 9, 9, 12, 12, 12, \ldots$	Bb^2	$9, 9, 9, 12, 12, 12, \ldots$
a^3	$6, 7, 8, 9, 10, 11, \ldots$	Aa^2	$7, 8, 8, 10, 11, 11, \ldots$
bab	$6, 6, 6, 9, 9, 9, \ldots$	Bab	$7, 7, 7, 10, 10, 10, \ldots$

QPI and Arctic Matrices

Theorem: for each QPI with slope 1 and period d, there is an equivalent ($=$ removes same rules) arctic interpretation of dimension $d \times d$.

example: $[A] : \begin{array}{c|c|c} 0 & 1 & 2 \\ \hline 2 & 4 & 5 \end{array}$, $\quad [A]'' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \boxed{1} & 1 \\ \boxed{0} & 0 & \boxed{1} \end{pmatrix}$

invariant: $[w](x) = d \cdot k + y \iff [w]''_{y,x} = k$.

$[a]'' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, [b]'' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [B]'' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Discussion

open questions:

- compare arctic power with standard power
- separate arctic dimensions
- arctic termination for TRS (via monotone algebra approach)

related work:

- Roberto Amadio: max plus quasi interpretations, TLCA 2003 (using polynomials $=$ convex piecewise linear functions)