Polynomial Bounds for $\mathbb{N}$-weighted word automata

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Derivational Complexity...

- domain D,
- size measure $|\cdot| : D \to \mathbb{N}$,
- (derivation) relation $\to \subseteq D^2$

height (derivational complexity) of $\to$:

$$n \mapsto \sup\{k \mid \exists s, t \in D : |s| \le n \wedge s \to^k t\}$$

...of (String) Rewriting

- $\{0 \to 1\}$ is linear
- $\{01 \to 10\}$ is quadratic (bubble sort)
- $\{01 \to 110\}$ is exponential

Rewriting and Weighted Aut.

- rewriting system R on Σ
- finite $(\mathbb{N}, +, \times)$-weighted automaton A on Σ

Idea: if $u \to_R v$, then $A(u) > A(v)$. This gives

- proof of termination of $\to_R$
- bound on derivational complexity of $\to_R$
  - in general: exponential
  - under certain conditions: polynomial

Monotone Algebras

The Σ-algebra of a $\mathbb{N}$-weighted automaton A

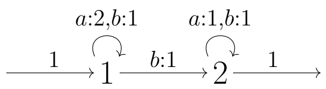
- domain: weight vectors $(Q(A) \to \mathbb{N})$
- order: $u > v : u(i) > v(i) \wedge \forall q : u(q) \ge v(q)$
- interpretation $[c]_A$: transition matrix of A for c

Prop: For A with $\forall c \in \Sigma, q \in \{i, f\} : [c]_A(q, q) > 0$, this algebra is monotone w.r.t. $>$.

Def: A is compatible with rewriting system R iff $\forall (l, r) \in R : [l]_A \ge [r]_A \wedge [l]_A(i, f) > [r]_A(i, f)$

Prop: Then, $A(w) = [w]_A(i, f)$ bounds number of R-steps from w.

Example

weighted automaton 

matrix interpretation $a \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

is compatible with
rewriting system $R = \{ab \to ba\}$, since

$ab \mapsto \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, ba \mapsto \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

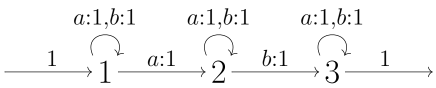
Note: automaton can be obtained as solution of a (diophantine) constraint system

Tight bounds

For $R = \{ab \to ba\}$, previous automaton is compatible, but not tight:

$[a^k b] = \begin{pmatrix} 2^k & 2^k \\ 0 & 1 \end{pmatrix}$ but $\mathrm{dc}_{\to_R}(a^k b) = k$

"better" automaton:



this interpretation is quadratically bounded
(the automaton exactly counts the inversions)

Upper triangular form

$m \in \mathbb{N}^{d \times d}$ is upper triangular (U) if
$\forall i, j : (i > j \Rightarrow m_{i,j} = 0) \wedge (i = j \Rightarrow m_{i,j} \in \{0, 1\})$
Example (previous slide):

$a \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

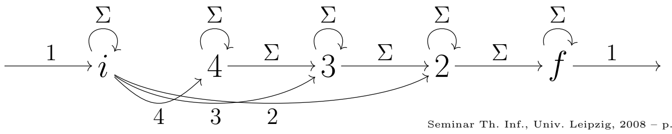
Prop: Let $[\cdot] : \Sigma \to U$. Then
$(n \mapsto \max\{[w]_{i,j} \mid w \in \Sigma^n\}) \in O(n \mapsto n^{\max(j-i, 0)})$.
Cor: upper triangular interpretation gives polynomial bound on derivational complexity
Note: easy modification of constraint system

Polynomial Derivations (Ex.)

$R_d = \{ki \to jk \mid j < k\}$ over $\Sigma = \{1, 2, \ldots, d\}$.
E.g. $R_2 = \{21 \to 12, \ldots\}$,
$R_3 = R_2 \cup \{31 \to 23, 32 \to 13, \ldots\}$
For $d \geq 2$, derivation with $\Theta(n^d)$ steps:

$$w = d^n(d-1)^n \ldots 1^n \to^* \text{reverse}(w)$$

compatible (upper triangular) $\mathbb{N}$-automaton (all weights are 1)

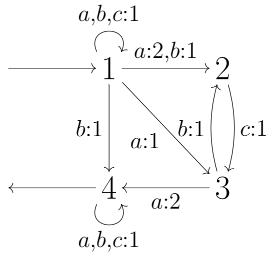


Other Matrix Forms

there are matrix interpretations with polynomial growth but not of upper triangular form. Example:

$$a \mapsto \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad c \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as weighted automaton:



Is $\mathbb{N}$-Automaton polynomial?

Decision procedure:

1. compute strongly connected components $A_1, \ldots, A_k$ of underlying graph.
2. if there is any arrow with weight > 1 inside one component, then growth is exponential.
3. from each component A_i, construct a (classical) automaton (all states initial and final)
4. if any A_i is ambiguous, then A is exponential.
5. Otherwise, A has polynomial growth.

Notes: degree is $<$ maximal number of SCCs on a chain of SCCs, this bound is not sharp.

Ambiguity

Def: A is non-ambiguous iff each $w \in L(A)$ has exactly one accepting path.
Thm: A is non-ambiguous iff

- the reduced form (all states reachable and productive)
- of $A \times A$ (cartesian product construction)
- consists of the main diagonal only.

(e.g. Sakarovitch: Theorie des Automates)

Constraints for Ambiguity

existentially quantified $R, P : Q^2 \to \{0, 1\}$

- $R(p, q)$: state $(p, q) \in Q \times Q$ is reachable
  $\forall p \in Q : R(p, p) \wedge \forall p_1, p_2, q_1, q_2 \in Q, c \in \Sigma :$
  $(R(p_1, q_1) \wedge p_1 \to_c p_2 \wedge q_1 \to_c q_2) \Rightarrow R(p_2, q_2)$
- $P(p, q)$: state $(p, q) \in Q \times Q$ is productive (similar)
- reduced automaton consists of diagonal only:
  $\forall p, q \in Q : R(p, q) \wedge P(p, q) \Rightarrow (p = q)$

$|Q|^2$ variables, $|Q|^4 \cdot |\Sigma|$ formula size

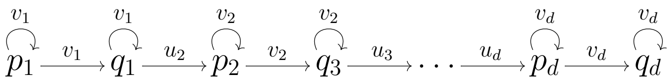
Constraints for SCCs

- $e : Q^2 \to \{0, 1\}$ "in the same SCC"
  let $p >_e q := (p > q) \wedge \neg e(p, q)$
  and $p \geq_e q := (p > q) \vee e(p, q)$.
- e symmetric, $>_e$ transitive, $\geq_e$ transitive,
  $\forall p, q \in Q : p \xrightarrow{w}_A q \wedge w > 1 \Rightarrow p >_e q$
  $\forall p, q \in Q : p \xrightarrow{w}_A q \wedge w \geq 1 \Rightarrow p \geq_e q$
- $d : Q \to \{0, \ldots, |Q| - 1\}$ for path length:
  $\forall p, q \in A : p \to_A q \Rightarrow d(p) \geq d(q)$
  $\forall p, q \in A : p \to_A q \wedge p >_e q \Rightarrow d(p) > d(q)$
- (loose) degree bound: $\forall q \in Q : d(q) \leq B$

A Sharp Bound (I)

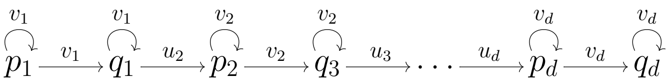
O.H. Ibarra, B. Ravikumar: *On Sparseness, ambiguity and other decision problems for acceptors and transducers*, STACS 1986.
A. Weber, H. Seidl: *On the degree of ambiguity of finite automata*, MFCS 1986, TCS 1991.
(cited in: Allauzen, Mohri, Rastogi: *General Algorithms for Testing the Ambiguitiy of Finite Automata*, 2008arXiv0802.3254A)

Thm: automaton contains graph



$\iff$ ambiguity is at least n^d.

A Sharp Bound (II)



- components can be encoded by $(p_i, p_i, q_i) \to^* (p_i, q_i, q_i)$ in $A \times A \times A$
- $q_i \to^* p_{i+1}$ is reachability in A

allows similar encoding as before (bound the length of chains of components)

Summary, Discussion

summary:

- polynomial $\mathbb{N}$-automaton growth is decidable
- can be encoded as FO-constraint system

open problems:

- is the method complete (or is there a polynomially bounded rewriting system that has no compatible polynomially bounded automaton)?
- is $\{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$ polynomial?
- generalize to tree automata, term rewriting

Related Work

$\mathbb{N}$-weighted word automaton $\equiv$ action of DT0L system on Parikh vectors

- L : Lindenmayer
- 0 : context-free, D : deterministic $\Rightarrow$ morphisms
- T : tabled $\Rightarrow$ several morphisms

direct correspondence between

- bounds for weights of automata
- bounds for (length) growth of DT0L systems