Size-Change Termination and Arctic Matrix Monoids

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WATA 2012, TU Dresden

Motivation/Summary

The *size change method* for automated termination analysis (Ben Amram et al, 2001):

- from input:
  - logic or functional program, term rewrite system, state transition system $R \subseteq \text{State} \times \text{State}$,
  - measure function (interpretation) $i : \text{State} \to \mathbb{N}^k$
- construct: set of arctic matrices M,
  - (expressing differences between measures)
- M^* universally unbounded $\Rightarrow$ R terminating.

The challenge is to decide unboundedness, or at least have a sufficient criterion.

Bounding the Changes

M change-bounds R iff

- R is a finite abstract rewrite system R (that is, a family of relations $\to_i$) on $\mathbb{N}^d$
- $M = \{M_i \mid i \in I\}$ is a set of arctic matrices with

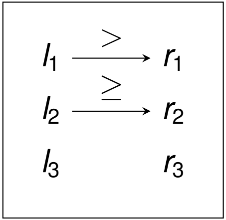
$$\forall i, x, y : x \to_i y \Rightarrow x \geq M_i \cdot y$$

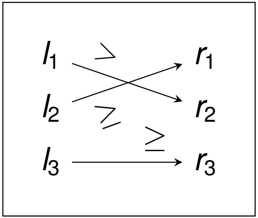
where

- arctic semiring $\mathbb{A} = (\{-\infty\} \cup \mathbb{Z}, \max, +, -\infty, 0)$
- relation $\geq$ is component-wise on $\mathbb{N}^d$

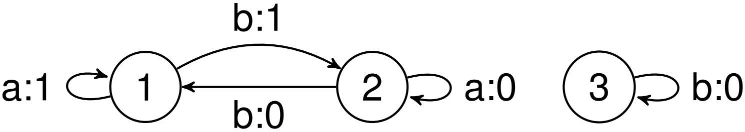
Bounding the Changes (Example)

$$F(x+1, y+1, z) \rightarrow F(x, y+1, F(y+1, x, z))$$





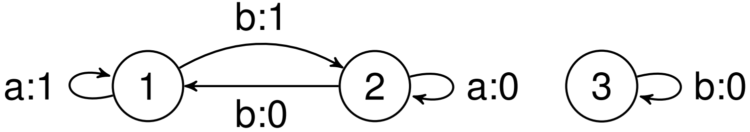
$$a = \begin{pmatrix} 1 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} \quad b = \begin{pmatrix} -\infty & 1 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$



The Basic Method (Theorem)

$M = \{M_i \mid i \in I\}$ is *universally tail-unbounded*:
$\forall u \in M^\omega : \exists i : \sup\{\|u_i \cdot \ldots \cdot u_j\| : j \geq i\} = +\infty$
(Norm of matrix is maximum of components.)

Ex: (map/plus, all states are initial and final)



Thm: If M is universally tail-unbounded
and M change-bounds R, then R is terminating.

Proof: $x \to_R^k y$ implies $|x| \geq \|u\| |y|$ for some
$u \in M^k$.

How to Find the Matrices

use domain-specific knowledge
(not the main topic of this talk)

- simple case, for programs with eager evaluation:
  vector of sizes of function arguments
- more general, for term rewriting:
  suitable ($=$ weakly monotone) vector-valued interpretation

Note: negative entries in change-matrices may be useful, correspond to *bounded increase*

Known Results on Unboundedness

Thm (Ben Amram et al.):
Universal tail-unboundedness is

- decidable (PSPACE-complete) over arctic naturals $\{-\infty\} \cup \mathbb{N}$,
  - (reduce to finite semiring $\{-\infty, 0, 1\}$)
- undecidable over arctic integers $\{-\infty\} \cup \mathbb{Z}$.
  - reduction from halting problem for two-counter machines
- decidable over arctive integers in special cases
  - *M* contains just one matrix (Bellman-Ford algorithm)
  - matrices in *M* have fan-in 1

Tail-Unboundedness and Loops

Def: looping(M) iff $\forall w \in M^+ : \exists e > 0 : w^e$ has some entry > 0 on main diagonal.

Thm: utu(M) $\iff$ looping(M).

Note: this is different from $(\mathbb{N}, +, \cdot)$.
For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$,
we have utu($\{A\}$) and $\neg$ looping($\{A\}$).

A Decision/Approximation Method

use classification $c : \mathbb{A} \to \{-\infty, 0, 1\}$ where

x	< 0	$= 0$	> 0
$c(x)$	$-\infty$	0	1

Properties:

- $c(A) \leq A$, $c(A) \cdot c(B) \leq c(A \cdot B)$
- for arctic integers:
  $\text{looping}(c(M)) \Rightarrow \text{looping}(M)$
- for arctic naturals:
  $\text{looping}(c(M)) \iff \text{looping}(M)$

M finite $\Rightarrow c(M)^*$ finite $\Rightarrow \text{looping}(c(M))$ decidable

Decision Method: Example

$$M_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & -1 \\ 0 & -3 \end{pmatrix},$$

$$c(M_1) = \begin{pmatrix} -\infty & 1 \\ -\infty & 1 \end{pmatrix}, c(M_2) = \begin{pmatrix} 1 & -\infty \\ 0 & -\infty \end{pmatrix},$$

$$c(\{M_1, M_2\}^+) \subseteq \begin{pmatrix} * & \geq 0 \\ * & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & * \\ \geq 0 & * \end{pmatrix}$$

is closed w.r.t. multiplication,
and each element is looping.

Improving the Approximation

for each $k \geq 1$: looping(M) $\iff$ looping(M^k).
we have $c(M)^k \leq c(M^k)$, possibly strict.

Example:

$$A = \begin{pmatrix} -\infty & 4 \\ -2 & -\infty \end{pmatrix}, c(A) = \begin{pmatrix} -\infty & 1 \\ -\infty & -\infty \end{pmatrix},$$

$$c(A)^2 = \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix}$$, thus $\neg$ looping($c(A)$).

$$A^2 = \begin{pmatrix} 2 & -\infty \\ -\infty & 2 \end{pmatrix}, c(A^2) = \begin{pmatrix} 1 & -\infty \\ -\infty & 1 \end{pmatrix}$$

so looping($c(A^2)$) and looping(A).

The Joint Spectral Subradius

Norm for arctic matrix $A \in \mathbb{A}^{d \times d}$:

$$\|A\| := \exp(\max_{i,j} A_{i,j})$$

joint spectral subradius of set M of matrices:

$$\mathrm{jssr}(M) := \inf \left\{ \|w\|^{1/k} \mid k > 0, w \in M^k \right\}$$

$\mathrm{jssr}(M) > 1 \Rightarrow \mathrm{utu}(M)$. The converse is false:

$$A = \begin{pmatrix} 0 & -\infty \\ -\infty & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -\infty \\ -\infty & -\infty \end{pmatrix}$$

$\mathrm{jssr}(\{A, B\}) = 1$, but $\mathrm{utu}(\{A, B\})$.

Extensions

domain-specific properties imply restrictions on sequences of steps (e.g., function calls)
$\Rightarrow$ consider only certain products of matrices.
from monoid ($=$ all products) go to category where

- objects $=$ abstract states,
- arrows $=$ sets of matrices.

corresponds to weighted automaton over semi-ring

- domain: sets of arctic matrices
- addition: union
- multiplication: component-wise

unboundedness is still decidable for arctic naturals, undecidable for arctic integers.

Remarks on Implementation

Given rewrite system R, want to find suitable interpretation i such that "size-change matrices" for i are tail-unbounded.

- standard approach: formulate all conditions as a constraint system, use SMT solver.
- problem: decision procedures for unboundedness are too hard (exponential, since they involve closure constructions)
- our proposal: polynomially sized constraint system for *candidates* (construct partial closure only), add separate search by bisection.