# Termination and Complexity
# *Lesson 1*
# Intl. School on Rewriting 2014

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# Day 1

- termination, complexity (abstractly)
  Hofbauer, Lautemann, RTA 98, http://www.theory.informatik.uni-kassel.de/~hofbauer/research/papers/RTA89-revised.pdf
- interpretations, matrix algebras
  Zantema: Termination of Rewriting, 2000,
  http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.19.2505
  Endrullis, Waldmann, Zantema: RTA 06,
  http://dx.doi.org/10.1007/11814771_47
- relative termination (lexicogr. combination) (abstract rewriting)
  Geser 1990: *Relative Termination* (cf. http://queuea9.wordpress.com/2010/05/27/getting-acquainted-with-relative-termination/)

# Termination

abstract rewriting system: relation $\to$ on a set $T$
relation $\to$ is terminating (or: well-founded) (or: stronly normalizing), notation $\mathrm{SN}(\to)$ iff there is no infinite sequence $[t_0, t_1, \ldots]$
(that is, the sequence is a mapping $t : \mathbb{N} \to T$)
with $\forall i : t_i \to t_{i+1}$
examples: $T = \mathbb{N}$,

- $\to_A = \{(x+1, x) \mid x \geq 0\}$
- $\to_B = \{(x, y) \mid x > y\}$
- $\to_B = \{(2x, 3x) \mid x > 0\}$

motivation: $T$ is set of machine states, $\to$ is one (non-deterministic) computation step, then "$\to$ is terminating" means "each computation will give a result" after a finite number of steps

# Derivational Complexity

with a function $|\cdot| : T \to \mathbb{N}$ (think "size"):
derivational complexity of $\to$ is the function
$\mathrm{dc}_{\to} : s \mapsto \sup\{k \mid |t_0| \leq s, t_0 \to^k t_k\}$
note: in general, $\mathrm{dc}_{\to} : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$
for terminating $\to$: $\mathrm{dc}_{\to} : \mathbb{N} \to \mathbb{N}$
motivation: quantitative bounds for computations (termination is qualitative bound)

# Monotone Interpretations

Definition: given $(T, \to_T)$ and $(D, \to_D)$, function $i : T \to D$ is monotone iff $\forall x, y \in T : x \to_T y \Rightarrow i(x) \to_D i(y)$.

Theorem:
if $i$ is monotone from $(T, \to_T)$ to $(D, \to_D)$, and $(D, \to_D)$ is well-founded, then $(T, \to_T)$ is well-founded.

typical application: $(D, \to_D)$ is $(\mathbb{N}, >)$.
we will consider different $D$ also, e.g., $D = \mathbb{N}^k$
(obvious question: what is well-founded $\to_D$ then? We'll see non-obvious answers.)

# Examples

from programming:

```
while (y > 0) { (x,y) := (y, mod(x,y));
```

interpretation: $i(x, y) = y$
from string rewriting (subword replacement in context)

- $R_1 = \{ab \to b\}$, $i(w) = |w|_a$
- $R_2 = \{ab \to ba\}$, $i(w) = |\{(p, q) \mid 1 \le p < q \le |w|, w_p = a, w_q = b\}|$
- $R_3 = \{aa \to aba\}$, ?
- $R_3 = \{ab \to baa\}$, ?

# Algebras

- interpretation: any mapping $\text{Term}(\Sigma) \to D$
- algebra: mapping defined by induction over term structure

Given signature $\Sigma$, a $\Sigma$-algebra $A$ consists of

- domain $D$ (any set),
- for each $f \in \Sigma$ with arity $k$, a $k$-ary function $f_A : D^k \to D$.

Then, each $t \in \text{Term}(\Sigma)$ has a value in the algebra, we can write $t_A$ or $A(t)$, or ...
Ex. (always: $\Sigma = \{+/2, 1/0\}$, domain $\mathbb{N}$)

- algebra $+_A(x, y) = x + y, 1_A = 0$
- algebra $+_A(x, y) = x - y, 1_A = 0$
- algebra $+_A(x, y) = x * y, 1_A = 0$

# Monotone Algebras

algebra is ordered if domain is ordered $(D, >)$.
is well-founded if $(D, >)$ is well-founded
ordered algebra is *monotone* if each function is monotone in each argument:
$d_i > d_i'$ implies
$f(\ldots, d_{i-1}, d_i, d_{i+1}, \ldots) > f(\ldots, d_{i-1}, d_i', d_{i+1}, \ldots)$
Ex.: which are monotone w.r.t. standard order $(\mathbb{N}, >)$?
(always $\Sigma = \{+/2, 1/0\}$)

- algebra $+_A(x, y) = x + y, 1_A = 0$
- algebra $+_A(x, y) = x - y, 1_A = 0$
- algebra $+_A(x, y) = x * y, 1_A = 0$

# Compatible Monotone Algebras

Def: a monotone $\Sigma$-algebra $A$ is *compatible* with a (rewrite) relation $\to$ on Term($\Sigma$):
if $s \to t$, then $A(s) > A(t)$.
Theorem: if $\to$ is compatible with a well-founded monotone algebra, then $\to$ is terminating.
Cor: derivation height of $t$ w.r.t. $\to$ is bounded by height of $D_a$ w.r.t. $>$.

# Polynomial Algebras

the classical case (in Termination, since 1970s), see also Baader/Nipkow

- algebra domain is $(\mathbb{N}, >)$
- algebra functions are polynomials

EXAMPLE

implication for complexity:

doubly exponential

shortcomings: cannot handle systems like

$\{fg \to ff, gf \to gg\}$.

$\{aa \to aba\}$ (since total termination implies simple termination)

# Algebras of Vectors (Def.)

note: $(\mathbb{N}, >)$ is total. We consider now monotone algebras

- for domain $\mathbb{N}^d$
- with non-total ordering $(x_1, x_2, \ldots, x_d) > (y_1, y_2, \ldots, y_d)$ iff $(x_1 > y_1) \wedge (x_2 \geq y_2) \wedge \ldots \wedge (x_d \geq y_d)$.

check that this is well-founded and non-total (both trivial)
example:
$[a](x_1, x_2) = (x_1 + x_2, 1), [b](x_1, x_2) = (x_1, 0)$.
check that these are monotone.
exercise: replace $\wedge$ by $\vee$, check properties.

# Algebras of Vectors (Appl.)

$[a](x_1, x_2) = (x_1 + x_2, 1), [b](x_1, x_2) = (x_1, 0)$.
check that $[aa](x_1, x_2) > [aba](x_1, x_2)$.
This algebra is compatible with $\{aa \to aba\}$, so it proves termination of that rewrite system.
Homework: find algebra (on $(\mathbb{N}^d, >)$) compatible with $\{fg \to ff, gf \to gg\}$.

# Matrix interpretations (Def.)

Def: an algebra on $(\mathbb{N}^d, >)$ is called matrix interpretation if each *k*-ary function symbol *f* is interpreted by a multlti-linear function $f_A : (\mathbb{N}^d) \to \mathbb{N}^d$ of shape
$f_A(x_1^T, \ldots, x_k^T) = M_1^{(f)} x_1^T + \ldots + M_k^{(f)} x_k^T + v^{(f)}$
where $M_i^{(f)} \in \mathbb{N}^{d \times d}$ (square matrices),
$v^{(f)} \in \mathbb{N}^{1 \times d}$ (column vector).

$$[a](x^T) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot x^T + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[b](x^T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x^T + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

# Matrix interpretations (Prop.)

- these multi-linear functions are closed w.r.t. composition
  example (repeated) compute
  $[a]([a](x^T)), [aba](x^T)$
- multi-linear function is monotone iff for each $M_i$, the top-left entry is $> 0$.
- these monotone multi-linear functions are closed under composition (of course)
- if $t \in \mathrm{Term}(\Sigma, X)$ and $\Sigma$-matrix interpretation $A$,
  then $t_A$ is $|\mathrm{Var}(t)|$-ary multi-linear function.

# Matrix interpretations (Prop.)

- Def: for $k$-ary multi-linear functions $f, g$, write $f > g$ iff $\forall x_i^T : f(x_1^T, \ldots, x_k^T) > g(x_1^T, \ldots, x_k^T)$
- Prop. $f > g$ iff $\forall i : M_i^{(f)} \geq M_i^{(g)}$ and $v^{(f)} > v^{(g)}$.
- Def: $A$ compatible with $R$: $\forall (l, r) \in R : l_A > r_A$,
  Thm: if $A$ monotone, and $A$ compatible with $R$, then $\to_R$ is terminating.

matrix interpretation is *certificate of termination* for $R$

- it implies the termination property
- its validity is easy to check

# Matrix Interpretation (easy examples)

linear polynomials are 1-dimensional matrix ints.
e.g., $[a](x) = 2x, [b](x) = x + 1$ is compatible with $ab \to ba$.
$[a](x) = 3x, [b](x) = x + 1$ is compatible with $ab \to bba$.

# Matrix Interpretation (hard examples)

- z001: (Zantema's Problem) $\{a^2b^2 \to b^3a^3\}$,
- z086: (Zantema's Other Problem) $\{ab \to c^2, ac \to b^2, bc \to a^2\}$
- These were contributed to TPDB (Termination Problems Data Base), `http://termination-portal.org/wiki/TPDB`
- by Hans Zantema, `http://www.win.tue.nl/~hzantema/`

# Matrix Interpretation (derivation lengths)

Prop. If $R$ admits matrix interpretation, then $\mathrm{dc}_R$ is at most exponential.
(Tomorrow: polynomially bounded matrix interpretations)
Exerc. find super-exonential derivations for $\{ab \to bba, cb \to bcc\}$.
Cor. This system does not admit matrix int. (but it is terminating—how do we prove it?)

# Combining well-founded relations

Definition: $(T_1, \to_1)$ and $(T_2, \to_2)$ define relation $\to$ on $T_1 \times T_2$ by
$(x_1, x_2) \to (y_1, y_2)$ iff $x_1 \to_1 y_1$ or ($x_1 = y_1$ and $x_2 \to_2 y_2$).
Notation $\text{lex}(\to_1, \to_2)$ for this $\to$.
Theorem: if $(T_1, \to_1)$ well-founded and $(T_2, \to_2)$ well-founded, then $(T_1 \times T_2, \text{lex}(\to_1, \to_2))$ well-founded.
Proof: by contradiction. Assume infinite $\to$-chain in $T_1 \times T_2$. First component must eventually be stationary.

# Combining well-founded relations (Application)

prove termination of $R \cup S$ where
$R = \{ab \to bba\}, S = \{cb \to bcc\}$.

# Combining well-founded relations (Outlook)

…so we can combine termination proofs. Can we combine statements about complexity? We will see tomorrow:

- in general, we get a huge bound
- in special cases, it is better