Termination and Complexity
Lesson 3
Intl. School on Rewriting 2014

Johannes Waldmann (HTWK Leipzig)

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Day 3

- exotic semirings and their matrix semirings
- arctic and tropical matrix int. for SRS
  Gebhardt, Waldmann: Act. Cyb. 19(2), http://www.inf.u-szeged.hu/actacybernetica/edb/vol19n2/Gebhardt_2009_ActaCybernetica.xml
- arctic and tropical matrix int. with DP method
  Koprowski, Waldmann, RTA 08, Act. Cyb. 19(2)
  http://www.inf.u-szeged.hu/actacybernetica/edb/vol19n2/Koprowski_2009_ActaCybernetica.xml
- matchbounded and deleting SRS, decomposition
  Endrullis, Hofbauer, Waldmann, WST 06
  http://www.imn.htwk-leipzig.de/~waldmann/talk/06/wst/decompose/

Exotic Multilinear Functions

Let E be any semiring. Consider multlti-linear functions $f_A : (E^d) \to E^d$ of shape (as before)
$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
For $E =$ arctic, tropical, fuzzy semiring:

- addition (max, min) is not monotone (examples?)
- multilinear functions are not monotone (except for very special cases)

Monotone Exotic Multilinear Functions

$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
need to remove (or treat specially) all additions:

- additions of vectors (the "+" in the above line)
- additions inside dot products in matrix-vector products

Monotone Exotic Matrix Multiplication

For $E =$ artic, tropical:
Define $>_0$ on E by
$x >_0 y \iff x > y \vee x = 0_E = y$
Define $>_0$ on E^d by
$(x_1 > y_1) \wedge (x_2 >_0 y_2) \wedge \cdots \wedge (x_d >_0 y_d)$
Question: find the set of matrices $M \subseteq E^{d \times d}$ such that
Prop. Multiplication by $A \in M$ is monotone w.r.t. $>$
Exercise: and show that this does not work for
$(x_1 > y_1) \wedge (x_2 \geq y_2) \wedge \cdots \wedge (x_d \geq y_d)$

Monotone Exotic Multilinear Functions

$f_a(x_1^T, \ldots, x_k^T) = M_1 x_1^T + \ldots + M_k x_k^T + v$
Prop. If f_A verifies

- $k = 1$ (unary symbols) and $v = (0_E, \ldots, 0_E)^T$ and top-left $(M_1) \neq 0_E$
- or $k = 0$ (nullary symbols)

then f_A is monotone w.r.t. $>_0$
Exercise: …and this does not hold if any of the conditions are violated.

Exotic Multilinear Functions and Compatibility

how to check that exotic matrix int. is compatible with rewrite rule?
By previous slide, it is enough to consider $f_a(x_1^T) = M_1 x_1^T$
Also recall that top-left$(M_1) \neq 0_E$.
Define $>_0$ on $E^{d\times d}$ by point-wise extension of $>_0$ on E.
Prop. $A >_0 B$ implies $Ax^T >_0 Bx^T$ for x with $x_1 \neq 0_E$.
example: $aa \to aba$ tropical with dim 3

Exotic Interpretations and Complexity

Thm. If R admits an arctic or tropical matrix interpretation, then dc_R is linear.

Top-Termination

Power of exotic matrix interpretations is much increased in combination with:
Dependendy Pairs Transformation:

- reduces a termination problem $\mathrm{SN}(R)$
- to a relative top-termination problem $\mathrm{SN}(\mathrm{DP}(R)_{\mathrm{top}}/R)$

because

- for top termination, interpretation need not be monotone at all
- for the “relative” part, need only be weakly monotone

Fuzzy Matrix Interpretations

the (min,max) semiring
since addition (min) is not monotone: remarks on arctic and tropical do apply: method cannot handle symbols of arity > 1
but does it work for unary? No:

- multiplication (max) is again not monotone.
- there are not enough values (height of interpretation is finite)

but

- this can be repaired (by transforming to tropical matrices)
- there is an efficient semi-algorithm to find fuzzy matrix int.

From Fuzzy to Tropical

consider only unary matrix interpretations of shape $f_A(x^T) = M \cdot x^T$

Def: from fuzzy matrix F, compute tropical matrix $T = \text{lift}_d(F)$

by point-wise multiplication by d.

Thm: Let A be a fuzzy matrix interpretation compatible with SRS R.

Let $m =$ max. finite entry in matrices in A and $w =$ max. length of rhs of R.

Then $\text{lift}_{m \cdot w}(A)$ (point-wise) is a tropical interpretation that is compatible with R.

(Fuzzy) Matrix I. as Weighted Autom.

given fuzzy matrix int. A compatible with SRS R, consider automaton (graph) on states $Q = \{1, \dots, d\}$, with 1 initial and final, and edge $p \stackrel{(a,h)}{\to} q$ iff $M_a(p, q) = h < +\infty$.

concatenate $p \stackrel{(a_1,h_1)}{\to} q \stackrel{(a_2,h_2)}{\to} r$ to $p \stackrel{(a_1 \cdot a_2, \max(h_1,h_2))}{\to} r$.

this automaton computes a function $A : Q \times \Sigma^* \times Q \to F : (p, w, q) \mapsto \min\{h | p \stackrel{(w,h)}{\to} q\}$

compatibility means: $\forall (l, r) \in R, p, q \in Q : A(p, l, q) >_0 A(p, r, q)$.

construction of compatible interpretation $\sim$ completion of automaton

Match-bounded string rewriting

for R over Σ, define $\text{match}_h(R)$ over $\text{lift}(\Sigma)$ by $\{l' \to \text{lift}_{h-1}(r) \mid \text{base}(l') = l, \max \text{height}(l) = h\}$
Ex.
$\text{match}_3(aa \to aba) = \{\ldots, a_3 a_0 \to a_2 b_2 a_2, \ldots\}$
For R-rewrite sequence $s : w_0 \to w_1 \to \ldots$
obtain $\text{match}_h(R)$-sequence s': start with $\text{lift}_h(w_0)$ and apply lifted rule at position.
Def: R is *match-bounded by h* if each s has a corresponding s' (i.e., labels do not go below 0).
Prop. R is match-bounded by $h \iff R$ admits a compatible fuzzy matrix interpretation with entries $\leq h$.

A Decomposition Result

new letters $\overleftarrow{a}$, $\overrightarrow{a}$,
operation on words $\overleftarrow{ab} = \overleftarrow{b}\,\overleftarrow{a}$, etc.
define
$E = \{a_x \overleftarrow{a_x} \to \epsilon \mid a \in \Sigma\} \cup \{\overrightarrow{a_x} a_x \to \epsilon \mid a \in \Sigma\}$
from each rule $(p a_h q \to r) \in \text{match}_h(R)$, where a_h is largest letter, construct rule $a_h \to \overleftarrow{p}\, r\, \overrightarrow{q}$, obtain set of rules C.
Prop. $\to_{\text{match}(R)} = \to_C \circ \to_E^* \cap (\text{lift}(\Sigma)^*)^2$
Thm. $\to_{\text{match}(R)}^* = \to_C^* \circ \to_E^* \cap (\text{lift}(\Sigma)^*)^2$
Note: $\to_C$ is subsitution, $\to_E$ is monadic, both are effectively REG-preserving
Cor. match(R) is REG-preserving
Cor. R match-bounded $\Rightarrow$ R REG-preserving

A completion procedure

- start with 1 state and loops $1 \stackrel{(a,h)}{\to} 1$
- while there is $(l, r) \in R, p, q \in Q$ such that $h = A(p, l, q) \not>_0 A(p, r, q)$:
  - pick $p', q' \in Q$ such that $p' \stackrel{(a,h)}{\to} q'$ is on path from p to q, where $l = u \cdot a \cdot v$.
  - add fresh path from p' to q' labelled $\overleftarrow{u} \cdot \text{lift}_{h-1}(r) \cdot \overrightarrow{v}$ (this is one $\to_C$-step)
  - close the automaton w.r.t. $\to_E$ (by adding ϵ-transitions)

Thm. this construction terminates iff R is match-bounded by h, the result is a fuzzy-weighted automaton compatible with R

Decision Problems for Exotic Int.

for E in natural, arctic, fuzzy semiring, consider these decision problems:

- given R (and d, and h), does R admit a comaptible E-interpretation with matrix size $\leq d$ and entries $\leq h$?

known results about complexity:

- d and h given: decidable (by enumeration)
- d given: decidable for fuzzy (can derive a bound for h)
- h given: decidable for fuzzy (by automata completion)