Weighted Automata and Rewriting Lecture 1

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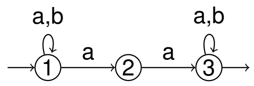
Motivation and Summary

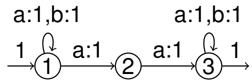
- Weighted automata (a.k.a. matrix interpretations) have become an essential tool for automated analysis of termination and complexity of rewriting
- A weighted automaton A evaluates its input (tree, string) in some semiring S, e.g., $(\mathbb{N}, 0, +, 1, \cdot)$, or $(\{-\infty\} \cup \mathbb{N}, -\infty, \max, 0, +)$
- If the valuation of A into well-founded $(S, >)$ is compatible with a rewrite system R, then this proves termination, and bounds derivational complexity, of $\to_R$
- R admits compatible valuation into $\mathbb{F} = (\mathbb{N} \cup \{+\infty\}, +\infty, \min, 0, \max)$ $\Rightarrow \to_R^*$ preserves regularity of languages

Example 1 ($\mathbb{F}$-weighted automaton)

- string rewriting system (SRS) $R = \{aa \to aba\}$ over alphabet $\Sigma = \{a, b\}$
- let us construct a $\mathbb{N}$-weighted Σ-automaton A with
  - A has a $(a, 2)$-loop at state 1
  - whenever there is a path $p \overset{aa}{\to} q$ with largest weight w_l, then there is a path $p \overset{aba}{\to} q$ with largest weight $w_r < w_l$
- Hints:
  - need two completion steps (add redex path)
  - second step can be improved (re-use path, fewer states)
- this A accepts $R^*(a^*)$, the R-many-step successors of a^*
- weights are actually from $\mathbb{F} = (\mathbb{N} \cup \{+\infty\}, \min, \max)$

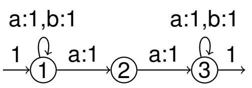
Example 2 ($\mathbb{N}$-weighted automaton)

- classical automaton 

  accepts/rejects a word, defines a language
- weighted automaton 

  defines a valuation $A : \Sigma^* \to \mathbb{N}$ by $A(w) =$ sum of weights of w-labelled accepting paths, weight of path $=$ product of its edge weights

$\mathbb{N}$-weighted Automata and Rewriting, Ex.

- automaton 

  $A(w) =$ number of occurences of aa in w
- string rewriting system $R = \{aa \to aba\}$ $w_0 = aba\underline{aa}b \to_R aba\underline{aba}b = w_1$,
- automaton and rewriting: $A(w_0) = 2 > 1 = A(w_1)$ this holds in general, so R terminates
- this lecture: make the above precise, and extend

References

- based on joint research (2002–present) with Jörg Endrullis, Alfons Geser, Dieter Hofbauer, Hans Zantema.
- overview and full references: J.W., *Automatic Termination*, RTA 2009.
- reference on weighted automata: Manfred Droste, Werner Kuich, and Heiko Vogler (Eds.), *Handbook of Weighted Automata*. Springer, 2009.
- slides for this course: `http://www.imn.htwk-leipzig.de/~waldmann/talk/17/isr/`

Semirings

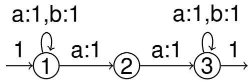
- Definition: $S = (D, 0, +, 1, \cdot)$ is semiring:
  - $(D, 0, +)$ is commutative monoid, $(D, 1, \cdot)$ is monoid
  - distributivity $\forall x, y, z \in D : x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$
  - $\forall x \in D : x \cdot 0 = 0 = 0 \cdot x$
- do not require:
  - commutativity of multiplication (because of matrices)
  - subtraction (ring), division (field)
- examples (used in this lecture)
  - standard (natural) semiring $(\mathbb{N}, 0, +, 1, \cdot)$
  - arctic semiring $\mathbb{A} = (\{-\infty\} \cup \mathbb{N}, -\infty, \max, 0, +)$
  - fuzzy semiring (not a standard name) $\mathbb{F} = (\{-\infty, +\infty\} \cup \mathbb{N}, +\infty, \min, -\infty, \max)$
- examples (used elsewhere) (Ex.: fill in missing pieces)
  - Booleans $\mathbb{B} = \{0, 1\}$, formal languages 2^{Σ^*}, relations $2^{U \times U}$.

Matrices over Semirings

- set of indices (later: states of automaton) Q
- matrix $m : Q \times Q \to S$, form a semiring
- Ex.: in $\mathbb{A} = (\{-\infty\} \cup \mathbb{N}, -\infty, \max, 0, +)$ $\begin{pmatrix} 0 & 5 \\ -\infty & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & -\infty \\ 1 & 2 \end{pmatrix} = ?$
- row and column vector: write as matrix $1 \times Q \to S, Q \times 1 \to S$
- Ex.: $\begin{pmatrix} 0 & 5 \end{pmatrix} \otimes \begin{pmatrix} 3 & 0 \\ -\infty & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = ?$

Weighted Automata — Def. (Pt. 1)

- S-weighted automaton A (alphabet Σ, states Q)
- *graphical approach*: directed graph on Q,
  edge labels from $\Sigma \times S$



- semantics defined via *runs* (paths)
- *algebraic approach*
  transition matrices $t_A : \Sigma \to (Q \times Q \to S)$
  $t(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, t(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$
- $t(a) \cdot t(b) \cdot t(a) =$ weights of paths labelled *aba*

Weighted Automata — Full Definition

S-weighted automaton A with alphabet Σ, states Q:

- transition matrices $t_A : \Sigma \to (Q \times Q \to S)$
- final weight vector (row) $f_A : 1 \times Q \to S$,
- initial weight vector (column) $i_A : Q \times 1 \to S$,

the semantics of A:

- define $t_A^* : \Sigma^* \to (Q \times Q \to S)$
  by $t_A^*[x_1, \dots, x_n] = t_A(x_1) \cdot \ldots \cdot t_A(x_n)$
- the weight function computed by A
  $A : \Sigma^* \to S : w \mapsto f_A \cdot t_A^*(w) \cdot i_A$

initial-final is confusing here (we want bottom-up in trees later)

Compatibility with Rewriting (Global)

- Def: $(S, >)$-weighted A
  is (globally) compatible with relation $\to$:
  $\forall x, y : x \to y \Rightarrow A(x) > A(y)$.
- Lemma: if $(S, >)$ is well-founded
  and A is compatible with $\to$,
  then $\to$ is well-founded (terminating).
- this is both obivous and impractical
  (how to check compatibility?)
- much more useful for rewrite relations $\to_R$:
  - local compatibility (check rules R instead of $\to_R$)
  - and monotonicity (extend to $\to_R$, apply contexts)

Monotonicity

- M a subset of matrices over S
  closed w.r.t. multiplication ($M^* \subseteq M$)
- with well-founded partial order $>_M$
  that is monotone w.r.t. left and right multiplication
  $\forall x, y, z \in M : x >_M y \Rightarrow xz >_M yz \wedge zx >_M zy$
  note: this implies $0 \notin M$
- example: $n \times n$-matrices over $\mathbb{N}$
  $M_1 = \{A \mid A_{1,1} \geq 1 \wedge A_{n,n} \geq 1\}$
  $A >_1 B$ iff $A_{1,n} > B_{1,n}$ and $\forall i, j : A_{i,j} \geq B_{i,j}$

Monotonicity – Examples

- which of these are closed w.r.t. multiplication?
  give the least restrictive monotone order for each.
- $n \times n$-matrices over $(\mathbb{N}, 0, +, 1, \cdot)$
  - $M_2 = \{A \mid A_{1,1} \geq 1\}$
  - $M_3 = \{A \mid A_{1,n} \geq 1\}$
  - $M_4 = \{A \mid \forall i : \exists j : A_{i,j} \geq 1\}$
- $n \times n$-matrices over $\mathbb{A} = (\{-\infty\} \cup \mathbb{N}, -\infty, \max, 0, +)$
  - $M_5 = \{A \mid A_{1,1} \geq 0\}$

Local Compatibility

- Def: $(S, >_S)$-weighted A admissible w.r.t. $(M, >_M)$:
  - $\forall x \in \Sigma : t_A(x) \in M$
  - $\forall m_1, m_2 \in M : m_1 >_M m_2 \Rightarrow f_A \cdot m_1 \cdot i_A >_S f_A \cdot m_2 \cdot i_A$
- Def: A locally compatible with R:
  $\forall (l, r) \in R : t_A^*(l) >_M t_A^*(r)$
- Thm: A admissible, A locally compatible with R
  and $(M, >)$ monotone
  $\Rightarrow A$ globally compatible with $\to_R$
- Cor: ... and $(S, >)$ well-founded
  $\Rightarrow \to_R$ well-founded.

Example 1

- $M = \{A \mid A_{1,1} \geq 1 \wedge A_{n,n} \geq 1\}$
  $A > B$ iff $A_{1,n} > B_{1,n}$ and $\forall i, j : A_{i,j} \geq B_{i,j}$
  $f = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$
- verify compatibility with $\{aa \to aba\}$ for

$$t(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, t(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- compute values for $aaab \to abaab \to ababab$

Example 1 – Remark

- use some CAS for calculations (e.g., `http://maxima.sourceforge.net/`)
-
```
a : matrix([1,1,0],[0,0,1],[0,0,1]) ;
b : matrix([1,0,0],[0,0,0],[0,0,1]) ;
[ a . a , a . b . a ];
```
-
```
               [ 1  1  1 ]  [ 1  1  0 ]
               [         ]  [         ]
(%o6)         [[ 0  0  1 ], [ 0  0  1 ]]
               [         ]  [         ]
               [ 0  0  1 ]  [ 0  0  1 ]
```

Example 2

- $M = \{A \mid A_{1,1} \geq 1 \wedge A_{n,n} \geq 1\}$
  $A > B$ iff $A_{1,n} > B_{1,n}$ and $\forall i, j : A_{i,j} \geq B_{i,j}$
- determine missing coefficients such that
  $t(a) = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}, t(b) = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix}$
  is compatible with $\{ab \to ba\}$

Example 2 – Remark

- use constraint language `http://smtlib.cs.uiowa.edu/standard.shtml`, and constraint solver (e.g., `https://github.com/Z3Prover/z3`)
-
```
(set-logic QF_NIA)
(set-option :produce-models true)
(declare-fun P () Int) (declare-fun Q () Int)
(declare-fun R () Int) (declare-fun S () Int)
(assert (and (< 0 P) (<= 0 Q) (< 0 R) (<= 0 S)))
(assert (> (+ (* P S) Q) (+ (* R Q) S)))
(check-sat) (get-value (P Q R S))
```
- `sat    ((P 14)  (Q 9)  (R 11) (S 7))`

Killer Example (2005): z086

- first termination proof of `SRS/Zantema/z086`
  $\{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$

$$a = \begin{pmatrix} 1&0&0&3&1 \\ 0&0&1&1&1 \\ 0&2&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{pmatrix} \quad b = \begin{pmatrix} 1&0&2&0&0 \\ 0&0&1&0&0 \\ 0&0&2&1&2 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{pmatrix} \quad c = \begin{pmatrix} 1&0&0&1&1 \\ 0&0&1&1&3 \\ 0&0&0&1&0 \\ 0&0&2&0&0 \\ 0&0&0&0&1 \end{pmatrix}$$

- found by solving the constraint system with a (hand written) bit-blaster and `http://minisat.se/`

Derivational Complexity

- derivation height (of term t, w.r.t. relation $\to$)
  $\text{dh}_\to(t) = \sup\{k \mid \exists t' : t \to^k t'\}$
- derivational complexity (of relation $\to$)
  $\text{dc}_\to(n) = \sup\{\text{dh}_\to(t) \mid \text{size}(t) \leq n\}$
- examples, where $\to$ is rewrite relation of SRS
  - $\text{dc}_{aa \to aba}$ linear
    $(aa)^k \to^k (aba)^k$, number of occurrences of aa
  - $\text{dc}_{ab \to ba}$ quadratic
    $a^k b^k \to^* b^k a^k$, number of inversions $a \dots b$
  - $\text{dc}_{ab \to baa}$ exponential
    $ab^k \to b^k a^{2^k}$. Upper bound?

Growth of Matrices

- Def: *growth* g_M of a set M of matrices
  is $k \mapsto \max\{\|m_1 \cdot \ldots \cdot m_k\| : m_i \in M\}$
  where $\|m\| = \max_{i,j} m_{i,j}$
- Def: growth g_A of $\mathbb{N}$-weighted automaton A
  is growth of its transition matrices
- Prop: A compatible with $\to_R$ implies $\text{dc}_{\to_R} \in O(\text{dc}_A)$.
- Ex., compatible with $\{ab \to ba\}$
  growth $\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ exponential

Polynomial Growth of Matrices

- Ex., compatible with $\{ab \to ba\}$
  growth $\left\{ \begin{pmatrix} 1&1&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0 \\ 0&1&1 \\ 0&0&1 \end{pmatrix} \right\}$ quadratic
- Thm: each $m \in M$ is upper triangular (below main diagonal: 0, on main diag: 0 or 1, above: $*$)
  $\Rightarrow$ growth M polynomial.
- Thm: (each SCC of M contains no > 1 and no diamond)
  $\iff$ growth M polynomial.
- *challenge problem (OPEN)*: polynomially growing matrix interpretation for $\text{z086} = \{a^2 \to bc, b^2 \to ac, c^2 \to ab\}$.