Sparse Tiling Through Overlap Closures for Termination of String Rewriting

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Preliminaries: Termination

- relation $\to$ is terminating (strongly normalizing) := there are no infinite $\to$-chains notations: $\mathrm{SN}(\to), \mathrm{SN}(\to_R), \mathrm{SN}(R)$.
- methods for proving termination of rewriting:
  - syntactical (precedence on symbols)
  - semantical (interprete symbols by functions over well-founded domain)
  - transformational ($\mathrm{SN}(R) \Leftarrow \mathrm{SN}(R')$)
- in particular: transformation that increases signature, give more room to pick predecence or interpretation
- ... by *tiling*: new signature consists of *tiles* (blocks of adjacent letters)

Preliminaries: Tiling

- $S = \{aa \to aba\}$ does not remove letters
- use tiles of width 2 (pairs of adjacent letters) $S_2 = \{[aa] \to [ab, ba]\}$, can simulate S-derivations S_2 removes letter $[aa]$: is terminating!
- in general: need (left and right) padding ex. from rule $ab \to ba$, create $[aa][ab][ba] \to [ab][ba][aa], [aa][ab][bb] \to [ab][ba][ab]$, $[ba][ab][ba] \to [bb][ba][aa], [ba][ab][bb] \to [bb][ba][ab]$
- instance: root labelling (Sternagel, Middeldorp, RTA 2008)
- our contribution:
  - use smaller set of tiles (for rewriting and for padding)
  - only those that appear in (certain) infinite derivations

Sparse Tiling: Definition and Motivation

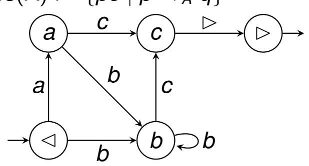
- Ex. the bordered 3-tiles of string $w = bbaab$ are $\mathrm{btiles}_3(w) = \{\lhd\lhd b, \lhd bb, bba, aab, ab\rhd, b\rhd\rhd\}$
- Def. [Zalcstein 1972] strictly locally testable language $\mathrm{Lang}(T) = \{w \mid \mathrm{btiles}(w) \subseteq T\}$
- this paper:
  - use such languages to over-approximate $R^*(L)$
  - represent T by finite automaton A,
  - ... constructed by completion
  - semantically label R by the partial algebra of A
  - ... to transform the termination problem of R on L.
  - *sparse*: T is the set of tiles that occur in rhs of forward closures (overlap closures, resp.)
- application: Matchbox wins Termcomp 2019 for SRS

Right-hand Sides of Forward Closures

- Def. $\mathrm{RFC}(R)$ = smallest set $M \subseteq \Sigma^*$ with
  - (start) $\mathrm{rhs}(R) \subseteq M$
  - (inner step) $(l, r) \in R \land ulv \in M \Rightarrow urv \in M$
  - (right extension) $(l_1 l_2, r) \in R \land ul_1 \in M \Rightarrow ur \in M$
- Thm. (Dershowitz 1981) R terminates on $\Sigma^* \iff R$ terminates on $\mathrm{RFC}(R)$
- Ex. $\mathrm{RFC}(\{ab \to ba\}) = b^+a$. Cor.: is terminating.
- Lemma: $\mathrm{RFC}(R) = (R \cup \mathrm{forw}(R))^*(\mathrm{rhs}(R))$ where $\mathrm{forw}(R) = \{l_1 \to_{\mathrm{Suffix}} r \mid (l_1 l_2 \to r) \in R\}$.
- Ex. $\mathrm{RFC}(\{ab \to ba\}) = \{ab \to ba, a \to_{\mathrm{Suffix}} ba\}^*(ba)$

Representing Sets of Tiles by Automata

- Def: the k-shift automaton (it remembers $k-1$ most recent letters read) alphabet $\Sigma \cup \{\rhd\}$, states $\mathrm{tiles}_{k-1}(\lhd^*\Sigma^*\rhd^*)$, initial state $\lhd^{k-1}$, final state $\rhd^{k-1}$, transitions: $p \xrightarrow{c}_A \mathrm{Suffix}_{k-1}(pc)$
- represents set of k-tiles $\mathrm{tiles}(A) := \{pc \mid p \xrightarrow{c}_A q\}$



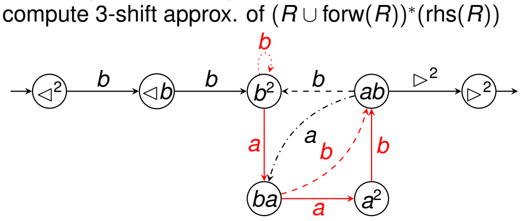
- Ex. 2-shift automaton $A =$ represents 2-tiles $\{\lhd a, \lhd b, ab, ac, bb, bc, c\rhd\}$ $\mathrm{Lang}(A) = (a+b)b^*c$

Rewrite Closure of Tiling Automata

- spec: given k-shift A, R over Σ, find k-shift A' over Σ s.t.
  - $\mathrm{Lang}(A) \subseteq \mathrm{Lang}(A')$
  - $u \in \mathrm{Lang}(A') \land u \to_R v \Rightarrow v \in \mathrm{Lang}(A')$
- implementation: when $(l, r) \in \mathrm{CC}_k(R)$ (right k-context closure) and $p \xrightarrow{l}_A q$, add transitions and states such that $p \xrightarrow{r}_A q$, until it stabilises
- by the k-shift property:
  - given p and r, the path $p \xrightarrow{r}_A q$ is fully determined, and it will indeed end in q
  - completion terminates since set of states is finite

Closure Example

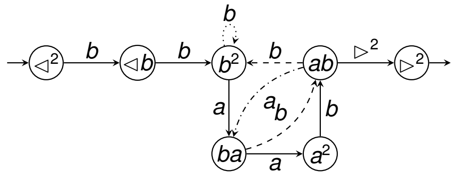
- for $R = \{ab^3 \to bbaab\}$, compute 3-shift approx. of $(R \cup \mathrm{forw}(R))^*(\mathrm{rhs}(R))$



- ... this is the path for $\mathrm{rhs}(R)$ $\to$ a redex for $(ab \to_{\mathrm{Suffix}} bbaab) \in \mathrm{forw}(R)$ dashed: new edges for corresponding reduct $\to$ a redex for $(ab \to_{\mathrm{Suffix}} bbaab) \in \mathrm{forw}(R)$ dotted: new edge for corresponding reduct $\to$ a redex for $(ab^3a \to bbaaba) \in \mathrm{CC}_1(R)$ dash-dotted: new edge for corresponding reduct

Semantic Labelling

- for $R = \{ab^3 \to bbaab\}$,



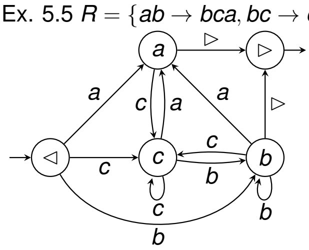
- semantically labelled R is $R' =$

$bba, bab, abb, b^3, bbx, bxy \to b^3, b^3, bba, baa, aab, abx, bxy$
$baa, aab, abb, b^3, bbx, bxy \to bab, abb, bba, baa, aab, abx, bxy$
$aba, bab, abb, b^3, bbx, bxy \to abb, b^3, bba, baa, aab, abx, bxy$

- $\mathrm{SN}(R')$ by weights $b^3 \mapsto 8, bab \mapsto 4, abb \mapsto 3, bba \mapsto 3$

Removing unreachable rules (Prop. 5.3)

- Ex. 5.5 $R = \{ab \to bca, bc \to cbb, ba \to acb\}$.



- $\mathrm{btiled}_T(ab \to bca) = \emptyset$ implies $\mathrm{SN}(R) \iff \mathrm{SN}(bc \to cbb, ba \to acb)$.
- we remove rule $ab \to bca$, even though A still contains redexes for $a \to_{\text{Suffix}} bca$.

Killer example: $a^2b^2 \to b^3a^3$

- Theorem: each paper on SRS termination contains a termination proof for Zantema's ($\approx$ 1993) problem
- Fact: as *z001*, it appears in the Termination Problems Data Base since the beginning of time ($=$ 2003)
- tiling for RFC; with semantic labelling (All), rule removal (Rem), weights (W); showing $(|R|, |\Sigma|)$ for each step:

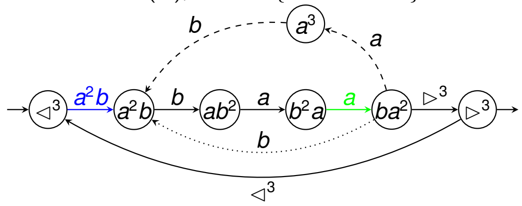
$$(1,2) \xrightarrow[\text{All}]{\text{RFC}_2} (4,4) \xrightarrow[\text{Rem}]{\text{RFC}_5} (3,4) \xrightarrow[\text{All}]{\text{RFC}_2} (12,8) \xrightarrow[\text{All}]{\text{RFC}_3} (105,26) \xrightarrow{\text{W}} (60,26)$$
$$\xrightarrow[\text{Rem}]{\text{RFC}_5} (37,26) \xrightarrow[\text{All}]{\text{RFC}_2} (97,44) \xrightarrow{\text{W}} (65,43) \xrightarrow[\text{Rem}]{\text{RFC}_5} (36,43) \xrightarrow{\text{W}} (28,43)$$
$$\xrightarrow[\text{All}]{\text{RFC}_2} (86,68) \xrightarrow{\text{W}} (50,62) \xrightarrow[\text{All}]{\text{RFC}_3} (246,128) \xrightarrow{\text{W}} (42,84)$$
$$\xrightarrow[\text{Rem}]{\text{RFC}_5} (2,44) \xrightarrow{\text{W}} (0,0)$$

Overlap Closures and Relative Termination

- Def: R terminates relative to S, notation: $\mathrm{SN}(R/S)$, if there is no $(R \cup S)$-derivation with infinitely many R steps. Ex: $\mathrm{SN}(aa \to aba / a \to aba)$.
- (recap) $\mathrm{SN}(R)$ iff $\mathrm{SN}(R)$ on $\mathrm{RFC}(R)$.
- (Ex. 6.1) $\mathrm{SN}(R/S)$ on $\mathrm{RFC}(R \cup S) \not\Rightarrow \mathrm{SN}(R/S)$. $R = \{ab \to a\}, S = \{c \to bc\}, \mathrm{RFC}(R \cup S) = a \cup b^+c$. But $abc \to_R ac \to_S abc$.
- Thm 6.7 $\mathrm{SN}(R/S)$ iff $\mathrm{SN}(R/S)$ on $\mathrm{ROC}(R \cup S)$. using right-hand sides of *overlap* closures
- apply left-recursive characterisation of ROC (overlap closure with rule) (see Appendix of paper).
- interesting case: (Cor 7.1.5) if $tx \in S$ and $yv \in S$ and $(xwy, z) \in R$, then $tzv \in S$

Example: Tiling for Overlap Closures

- 4-tiles for $\mathrm{ROC}(R)$, for $R = \{a^3 \to a^2b^2a^2\}$.



- if $tx \in S$ and $yv \in S$ and $(xwy, z) \in R$, then $tzv \in S$
  x is path to final state (since $x \in \mathrm{Suffix}(S)$)
  y is path from initial state (since $y \in \mathrm{Prefix}(S)$)
  use rewrite rule with border letters: $x \triangleright^{k-1} \triangleleft^{k-1} y \to z$
  Ex: $aaa \cdot ab \to a^2b^2a^2 \cdot ab$, reduct needs dashed edges

Implementation, Experiments, Questions

- implemented as part of termination prover `https://gitlab.imn.htwk-leipzig.de/waldmann/pure-matchbox`
- performance, including *Termcomp 2019* (SRS)

Relative		matrices no	matrices yes
tiling	no	1	72
tiling	yes	176	*225*

Standard		MB, DP, matr. none	MB, DP, matr. all
tiling	no	100	1122
tiling	yes	512	*1133*

- ? better proof search strategy for SRS Standard
- ? sparse tiling for TRS (RFC needs linearity)
- ? relation between matchbounds and tiling
- ? relation between tilings of different widths