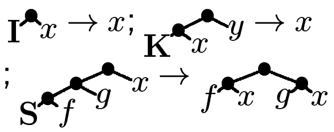
## *Kombinatorische Logik*

Namen für Funktionen

- Identität $I(x) = x$
- Konstanz $K(x, y) = x$
- vertauschen $C(f)(x, y) = f(y, x)$
- verdoppeln $D(f)(x) = f(f(x))$
- einsetzen $B(f, g)(x) = f(g(x))$
- verteilen $S(f, g)(x) = f(x)(g(x))$

binäre Bäume (= Terme)

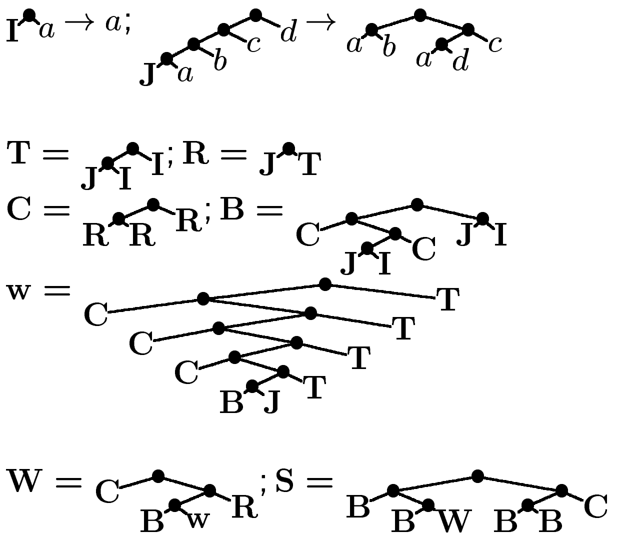
- innere Knoten: Applikation 
- Blätter: Kombinatoren $\mathbf{I}$, $\mathbf{K}$, ...

Ersetzungsregeln 

$\Rightarrow$ Term–Ersetzungs–System



## *Die Basis* $\{\mathbf{I}, \mathbf{J}\}$

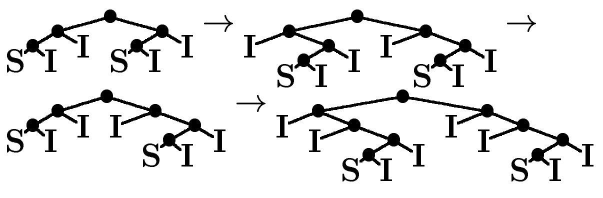


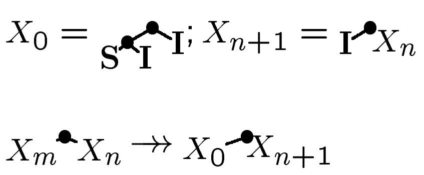


## *Termination (Beispiele)*

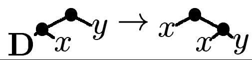
In $\mathbf{CL}((\mathbf{I}, \mathbf{K}, \mathbf{B}, \mathbf{C}))$ sind alle Reduktionsketten endlich.

In $\mathbf{CL}((\mathbf{S}, \mathbf{I}))$:

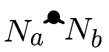




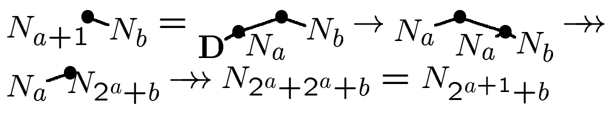


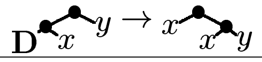
## *Termination für* 

Normalformen $N_0 = \mathbf{D}$; $N_{k+1} =$ 

Satz:  hat Normalform $N_{2^a+b}$, erreicht diese in $2^a - 1$ Schritten.

Beweis: 



## *Norm für* 

$\|\mathbf{D}\| = 1$; $\|(X\,Y)\| = 2^{\|X\|-1} + \|Y\|$

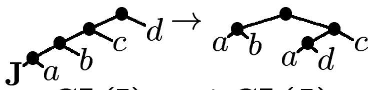
Satz: $X \to Y \Rightarrow \|X\| = \|Y\|$

$\mathsf{s}(\mathbf{D}) = 0$; $\mathsf{s}\big((X\,Y)\big) =$

$2 \cdot \mathsf{s}(X) + \mathsf{s}(Y) + 2^{\|X\|-1} - 1$

Satz: $X \to Y \Rightarrow \mathsf{s}(X) > \mathsf{s}(Y)$

Probst und Studer (2000): ähnlich für



$\Rightarrow$ $\mathbf{CL}(\mathbf{I})$ und $\mathbf{CL}(\mathbf{J})$ terminieren,

aber $\mathbf{CL}((\mathbf{I},\mathbf{J}))$ nicht!



## *Maßzahlen für Kombinatoren*

- Anzahl der eßbaren Argumente

max $\{k \mid X v_1 \dots v_k \to^* Y \to_{\mathsf{top}} Z\}$

(Visser 1981, van Oostrom 2000)

$X \in \mathbf{CL}(\mathbf{S})$ konsumiert $\leq 3$ Argumente

- nie aktivierte Argumente

max $\{k \mid X v_1 \dots v_k \not\to^* Y \rhd (v_i\,*)\}$

$X \in \mathbf{CL}(\mathbf{D})$ ignoriert $\leq 3$ Argumente

In $\mathbf{CL}(\mathbf{S})$: betrachte

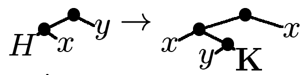
$\{X \mid \ X$ ignoriert 1 Argument$\}$



## *Wortprobleme für* $\mathbf{CL}()$ *(R. Statman)*

kombinat. vollst. $\Rightarrow$ unentscheidbar

Satz: Jede vollst. Basis enthält
3–Kombinator.
Vermutung: Wortproblem für
2–Kombinatoren entscheidbar.

Definition (Bsp): 
ist *impure* Kombinator.

Satz: Wortproblem für *impure*
1–Kombinatoren entscheidbar
(Beweis: reguläre Baumsprachen!)
. . . 2–Kombinatoren unentscheidbar!

Verbindung zu 2–Variable–Logiken?



## *Fixpunkte von Morphismen (Beispiel)*

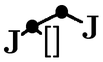
$h : \Sigma^* \to \Sigma^* : R \mapsto RL, L \mapsto RR$
$h(u \cdot v) = h(u) \cdot h(v)$

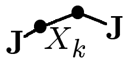
Def: $N_k = h^k(R)$; $N_\omega = \lim_{k\to\infty} N_k$
Bsp: $N_2 = RLRR, N_3 = RLRRRLRL$
Satz: $R \sqsubset h(R) \Rightarrow (i < j \Rightarrow N_i \sqsubset N_j)$

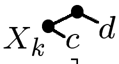
Def: $\overline{wR} = wL, \overline{wL} = wR$
Bsp: $\overline{N_2} = RLRL, \overline{N_3} = RLRRRLRR,$
Satz: $N_{k+1} = N_k\overline{N_k}$

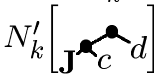
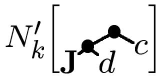
Def: $wR' = w, wL' = w$
Bsp: $N'_2 = RLR, N'_3 = RLRRRLR$
$N'_{k+1} =$ if $2 \mid k$ then $N'_k R N'_k$ else $N'_k L N'_k$

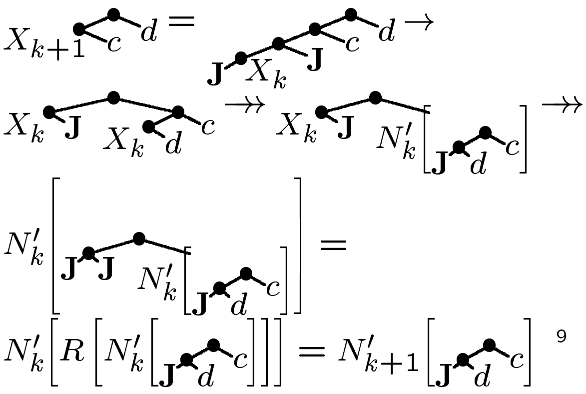
## *Normalformen in* $\mathbf{CL}(\mathbf{J})$

Kontexte $L[] =$ ; $R[] =$ 

$X_0 = \mathbf{J}; X_{k+1} = L[X_k] =$ 

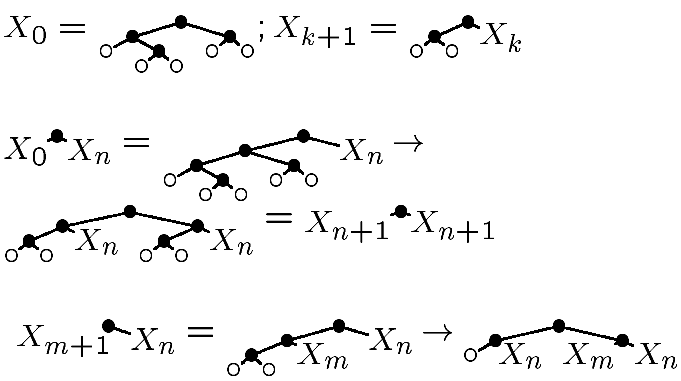
Normalform von  ist

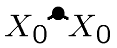
if $2 \mid k$ then  else 

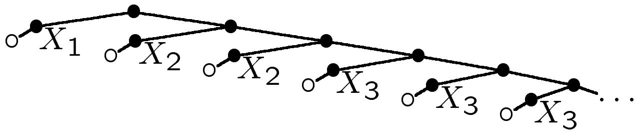




## *Unendliche Normalformen in* $\mathbf{CL}(\mathbf{S})$



Limes von  ist





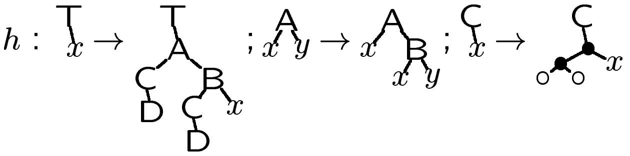
## *HD0L–(Baum–)Sprachen*

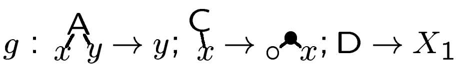
$w = ABABBABBBABBBB\ldots$

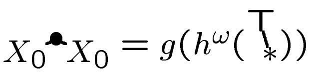
$h : T \mapsto TAB, A \mapsto AB, B \mapsto B$

$T \mapsto TAB \mapsto TAB\ AB\ B \mapsto TAB\ AB\ B\ AB\ B\ B \mapsto \ldots$

$g : T \mapsto \epsilon, A \mapsto A, B \mapsto B\ ;\ w = g(h^\omega(T))$





lim 



## *Wortproblem und HD0L–Sprachen*

Wortproblem: gilt $X \leftrightarrow^* Y$ ?
bei Konfluenz: $\exists Z : X \rightarrow^* Z \leftarrow^* Y$ ?
bei Termination: gilt $\mathsf{nf}(X) = \mathsf{nf}(Y)$ ?

L : Aristid Lindenmayer (1968)
0 : kontextfrei, D : deterministisch
(Fixpunkt eines Homomorphismus)
H : homomorphes Bild davon

Satz: Äquivalenz von HD0L–Systemen ist entscheidbar. Beweis:
1. Ehrenfeuchts “Vermutung”
2. Algorithmus von Makanin