Normalization of $\mathbf{S}$-Terms is Decidable

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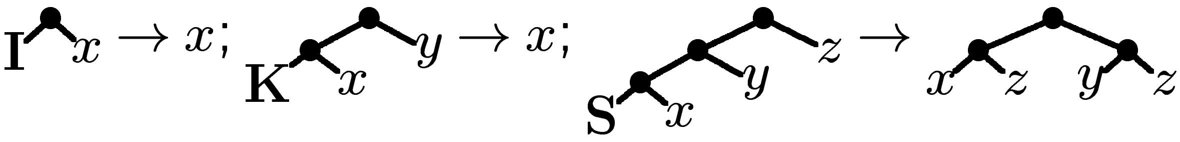
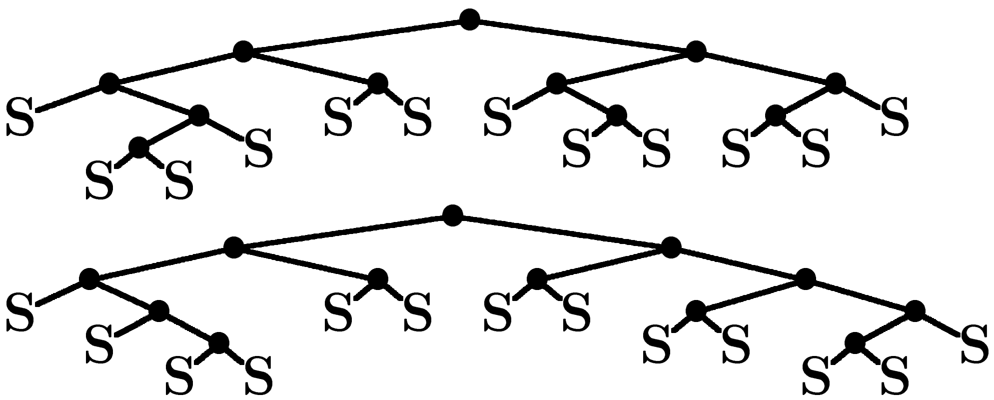


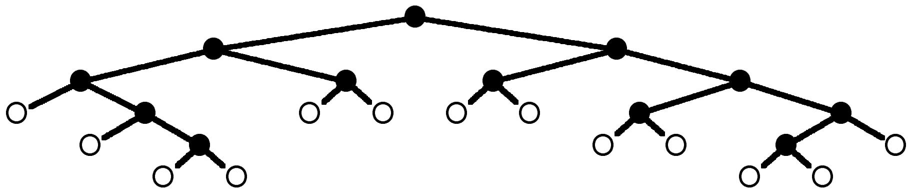
$\mathbf{CL}\,(\mathbf{S},\mathbf{K})$ is combinatorially complete.
Properties of $\{\mathbf{S},\mathbf{K}\}$-terms are undecidable:
normalization, word problem.

$\mathbf{CL}(\mathbf{K})$ is strongly normalizing.
Barendregt, Bergstra, Klop, Volken 1975:
How difficult is $\mathbf{CL}(\mathbf{S})$?

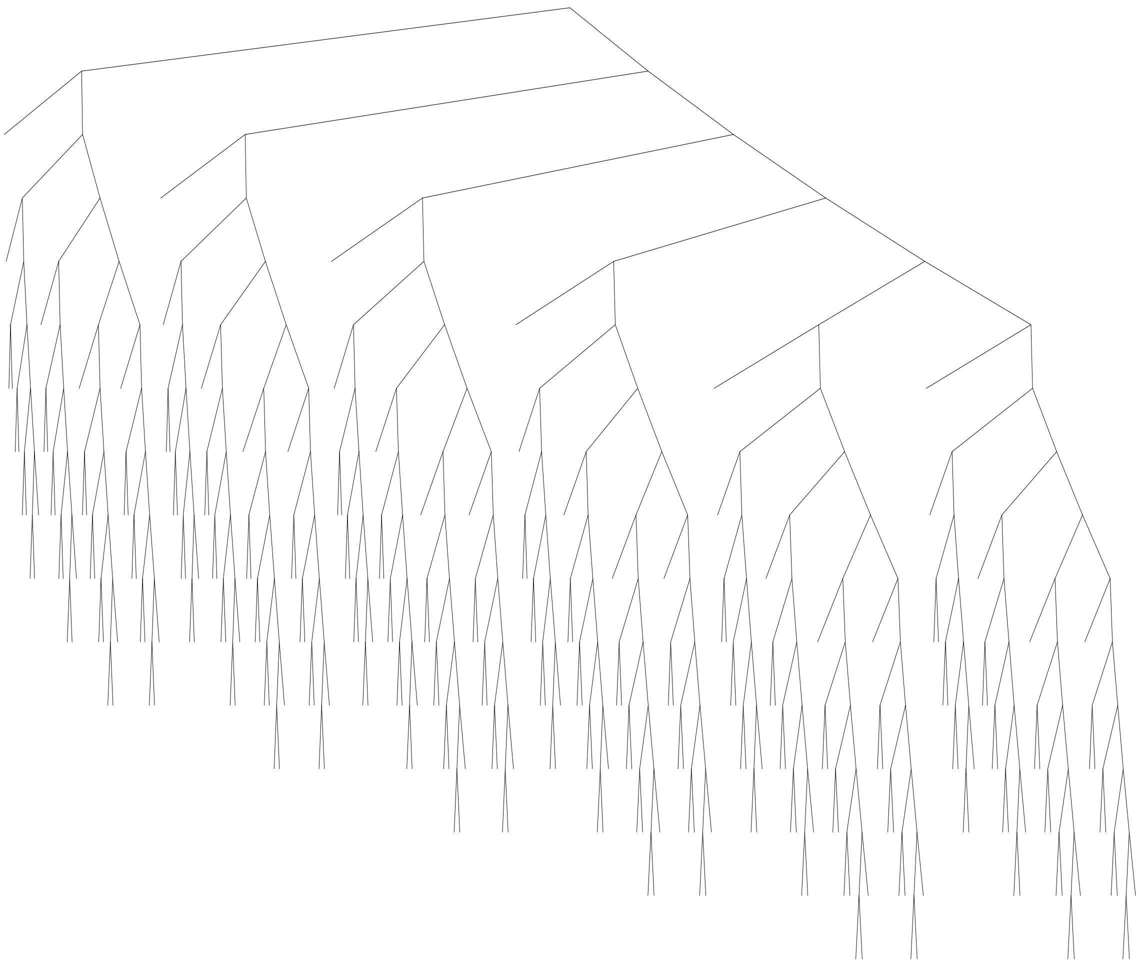
orthogonal, confluent,
non-erasing, term is WN iff SN

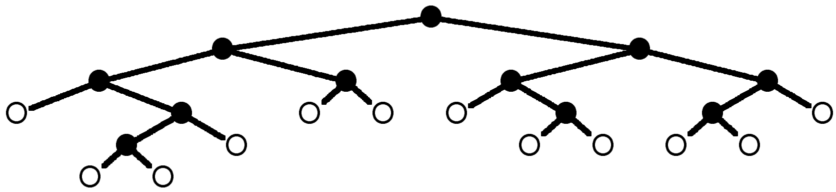
Do these terms normalize?



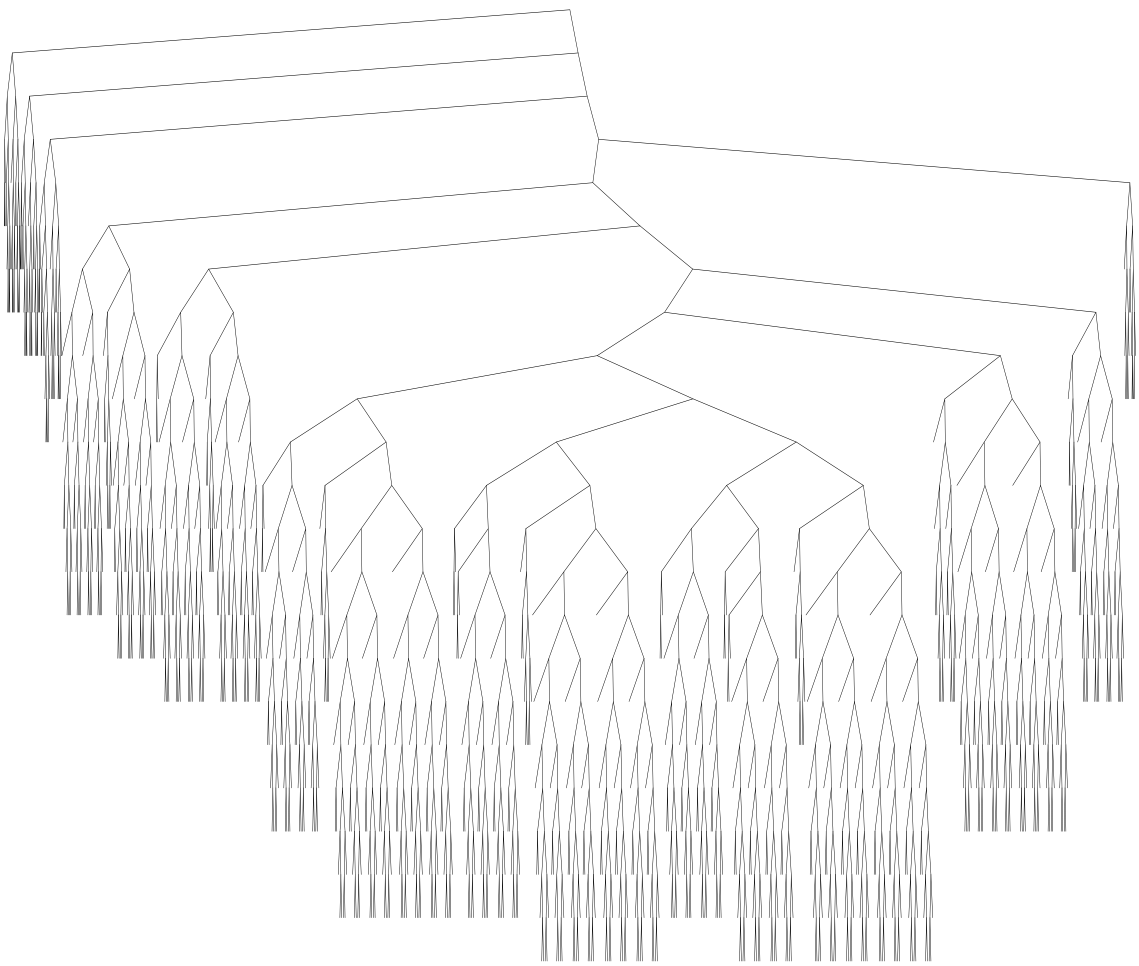


has normal form:



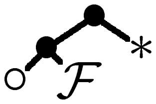
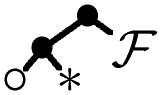


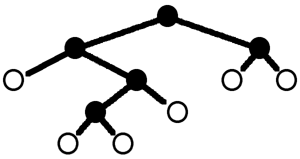
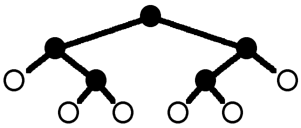
has no normal form. After 30 steps:



A Rational Set of Non-Normalizing Terms

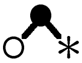
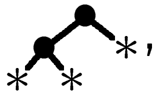
Def. $\mathcal{Q}_1 = \mathcal{M} \setminus \circ$, $\mathcal{Q}_2 = \mathcal{M} \setminus \{\circ,$  $\}$

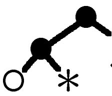
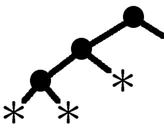
Def. $\mathcal{F} \to$  $\cup$  $\cup$ $\mathcal{Q}_2$  $\mathcal{Q}_1$

Ex.  $\in \mathcal{F}$,  $\in \mathcal{F}$,

on next slide: show $\mathcal{F}$  $\mathcal{F} \not\twoheadrightarrow \infty$.

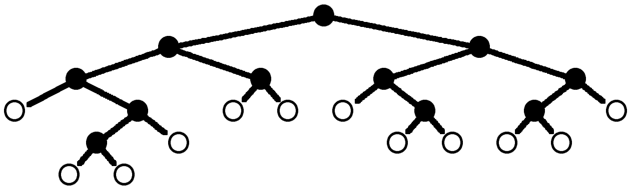
Lemma. $\mathcal{Q}_1$  $\mathcal{F} \subseteq \mathcal{F}$

Proof. $\mathcal{Q}_1 =$  $\cup$ ,

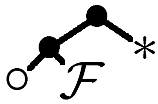
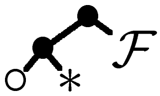
 $\mathcal{F} \subseteq \mathcal{F}$,  $\mathcal{F} \subseteq \mathcal{Q}_2$  $\mathcal{F} \subseteq \mathcal{Q}_2$  $\mathcal{Q}_1 \subseteq \mathcal{F}$.

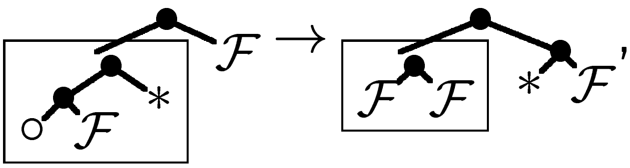
$\square$

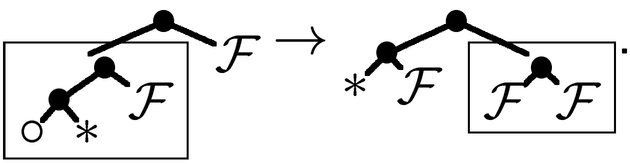
Prop. If $X \in \mathcal{F}\overset{\bullet}{\wedge}\mathcal{F}$, then $X \twoheadrightarrow \infty$

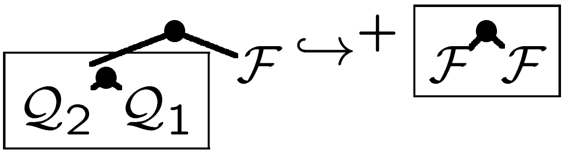
Ex.  $\twoheadrightarrow \infty$

Proof.

Recall $\mathcal{F} =$  $\cup$  $\cup\ \mathcal{Q}_2\overset{\bullet}{\wedge}\mathcal{Q}_1$.







see next slide

Recall: want to show $\mathcal{F} \bullet \mathcal{F} \twoheadrightarrow \infty$,
case $(\mathcal{Q}_2 \bullet \mathcal{Q}_1) \bullet \mathcal{F} \twoheadrightarrow \infty$.

Let $(X \bullet Y) \bullet Z \in (\mathcal{Q}_2 \bullet \mathcal{Q}_1) \bullet \mathcal{F}$.

If $X \twoheadrightarrow \infty$, then $(X \bullet Y) \bullet Z \twoheadrightarrow \infty$.

Otherwise X has nf.

Recall $\mathcal{Q}_1 = \mathcal{M} \setminus \circ$, $\mathcal{Q}_2 = \mathcal{M} \setminus \left\{\circ, \circ \bullet \circ\right\}$.

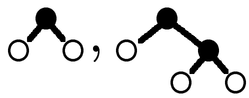
$X \twoheadrightarrow \mathcal{Q}_2 \cap \mathcal{N} \subseteq \circ \bullet \mathcal{Q}_1 \ \cup \ (\circ \bullet *) \bullet *$.

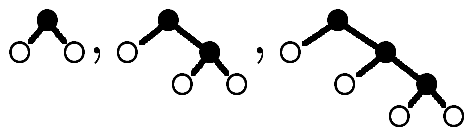
$$\left(\boxed{\circ \bullet \mathcal{Q}_1} \bullet \mathcal{Q}_1\right) \bullet \mathcal{F} \to (\mathcal{Q}_1 \bullet \mathcal{F}) \bullet (\mathcal{Q}_1 \bullet \mathcal{F}) \subseteq \mathcal{F} \bullet \mathcal{F}$$

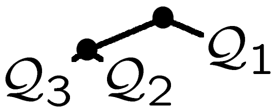
$$\left(\boxed{(\circ \bullet *) \bullet *} \bullet \mathcal{Q}_1\right) \bullet \mathcal{F} \to \left((* \bullet \mathcal{Q}_1) \bullet (* \bullet \mathcal{Q}_1)\right) \bullet \mathcal{F} \subseteq (\mathcal{Q}_2 \bullet \mathcal{Q}_1) \bullet \mathcal{F} \subseteq \mathcal{F} \bullet \mathcal{F}$$

$\square$

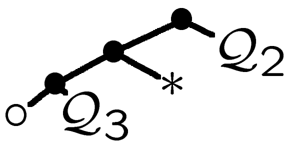
More Rational Non-Normalizing Sets

Def. $\mathcal{Q}_3 = \mathcal{M} \setminus \{\circ,$  $\}$

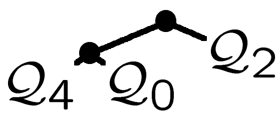
Def. $\mathcal{Q}_4 = \mathcal{M} \setminus \{\circ,$  $\}$

Prop. 

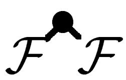
$\twoheadrightarrow \infty$ $\square$

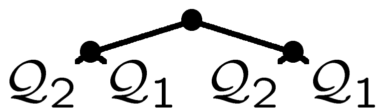
Prop. 

$\twoheadrightarrow \infty$ $\square$

read this as (nearly) 

$\twoheadrightarrow \infty$

remember 

$=$ 

$\twoheadrightarrow \infty$

The sum of the indices is six. *Why?*

Rational Sets of Normalizing Terms

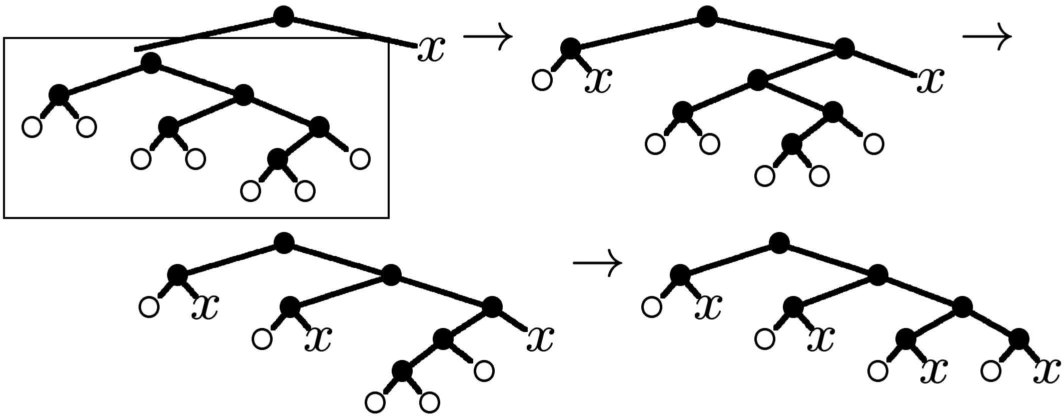
Def. *function* *argument*, *active* *passive*

Def. $\mathcal{N}_0$ such that $\downarrow\left(\mathcal{N}_0\, x\right)$ and $\mathcal{N}_0\, x \not\hookrightarrow x\, *$

(no activation)

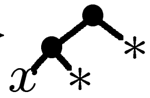
Prop. $\mathcal{N}_0 \to \circ \;\cup\; \circ\,\mathcal{N} \;\cup\; (\circ\,\circ)\,\mathcal{N}_0$ $\square$

Ex.



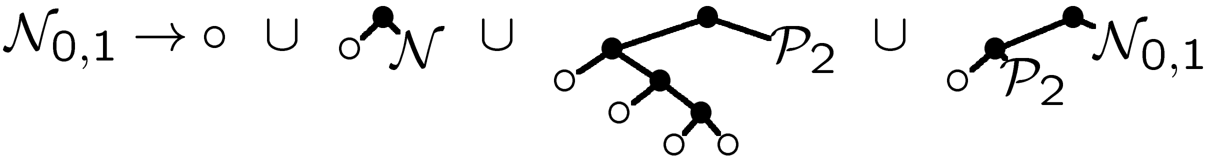
Prop. $\downarrow\left(\mathcal{N}_0\,\mathcal{N}\right)$ $\square$

Def. $\mathcal{N}_1$ such that for $\mathcal{N}_{0,1} = \mathcal{N}_0 \cup \mathcal{N}_1$:

$\downarrow\left(\mathcal{N}_{0,1}\, x\right)$ and $\mathcal{N}_{0,1}\, x \not\twoheadrightarrow$ 

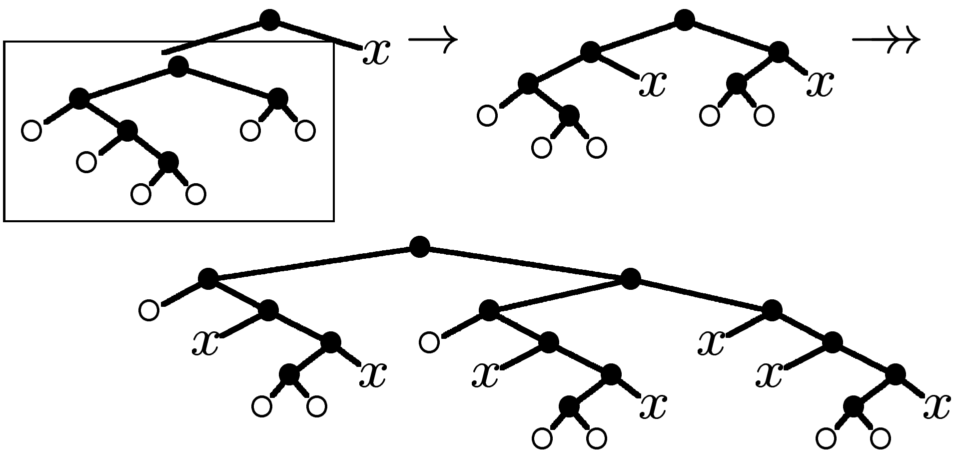
(activate, but supply one argument at most)

Prop.



where $\mathcal{P}_2 = \{\circ,$  $\}$ $\square$

Ex.

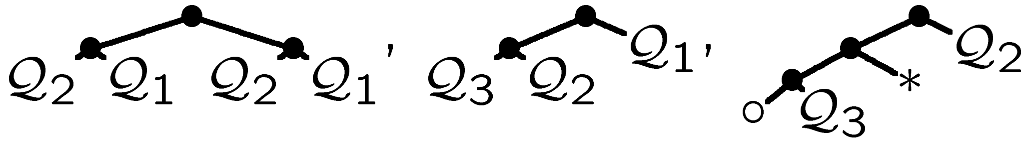


Prop. $\downarrow\left(\mathcal{N}_{0,1}\, \mathcal{N}_0\right)$

Proof. Apply previous proposition to redexes $\mathcal{N}_0\, *$. $\square$

Deciding Normalization

not normalizing:



normalizing: $\mathcal{N}_0\,\mathcal{N}$, $\mathcal{N}_{0,1}\,\mathcal{N}_0$

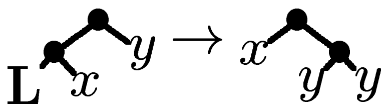
Assume $\mathcal{N}\,\mathcal{N}$. remaining: $\mathcal{N}_1\,\mathcal{N}_1$
(see paper, use $\mathcal{N}_1 = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$)

and $\mathcal{N}_2\,\mathcal{P}_2$ (do some reductions
to reach one of the above cases)

Thm. Normalization of $\mathbf{S}$-terms is decidable.

Proof. Recursively check normalization
for left and right subterm of the root.
Compute normal forms,
then apply the case distinction from above. $\square$

Question.
Is the word problem decidable for $\mathbf{CL(S)}$?

It is for $\mathbf{CL(L)}$ with 

(Statman 1988)

Question.
Is the set of normalizing $\mathbf{S}$-terms rational?

Conj. Yes.
It has a minimal deterministic automaton with 38 states and 1002 rules.

Find a top-down proof and construction (cf. Thomas Genet's paper, this RTA).