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Busy Beaver PCPs

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PCP basics

Post's correspondence problem:

instance: $\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix}$

solution: $i_1, i_2, \dots, i_n$
with $u_{i_1} \cdot u_{i_2} \cdot \dots \cdot u_{i_n} = v_{i_1} \cdot v_{i_2} \cdot \dots \cdot v_{i_n}$

example: $\begin{pmatrix} 110 & 0 & 1 \\ A & B & C \\ 1 & 1 & 01 \end{pmatrix}$

solution: $AACBCCBC$ because

$$\begin{array}{rcl} u_A u_A u_C u_B u_C u_C u_B u_C & = & v_A v_A v_C v_B v_C v_C v_B v_C \\ 110\ 110\ 1\ 0\ 1\ 1\ 0\ 1 & = & 1\ 1\ 01\ 1\ 01\ 01\ 1\ 01 \end{array}$$

Solvability of instances is undecidable.

Emil Post. *A variant of a recursively unsolvable problem.* Bull. AMS, 52:264–268, 1946.

PCP subclasses

$$\text{size}\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix} := s$$

PCP(s) := all instances of size $\leq s$

PCP(7) is undecidable. — Y. Matiyasevich and G. Sénizergues, LICS 1996

PCP(2) is decidable. — A. Ehrenfeucht, J. Karhumaki, and G. Rozenberg. TCS 21(2):119–144, 1982.

what about PCP($3 \dots 6$)?

width := $\max\{|u_i|, |v_i| : 1 \leq i \leq s\}$

PCP(s, w) := PCP(s) with width $\leq w$.

collect data from PCP($3 \dots 4, 3 \dots 4$)

Hunting for Busy PCP Beavers

What?

BB: instance is *small*, shortest solution is *long*

Why?

hope to find computable f
such that $\forall w : \forall I \in \text{PCP}(3, w)$:
$\text{Sol}(I) \neq \emptyset \iff \exists s \in \text{Sol}(I) : |s| \leq f(w)$

this would prove PCP(3) decidable

How?

main problem: prune the search tree, to

- find solutions quickly
- reject unsolvable instances early

Tricks of the Trade (example)

prove $\begin{pmatrix} 101 & 1 & 0 \\ A & B & C \\ 0 & 101 & 1 \end{pmatrix}$ unsolvable:

starts $\begin{array}{l} 1\,0\,1\,1 \\ 101\,1\,101\,101 \end{array}$, ends $\begin{array}{r} 1\,1\,0\,1 \\ 101\,101\,1\,101 \end{array}$

upper side must overtake lower side $\Rightarrow$ use A

but differences are always even

here's a challenge: prove

$\begin{pmatrix} 0 & 10 & 001 \\ 001 & 0 & 1 \end{pmatrix}$ unsolvable!

(collect prize from me, during WST+RTA)

Counting

$$\begin{pmatrix} 100 & 0 & 10 \\ A & B & C \\ 0 & 100 & 100 \end{pmatrix}$$

count balances of letters:

$$\begin{array}{rcl} 0: \; a-b-c &=& 0 \\ 1: \; a-b &=& 0 \end{array}$$

$\Rightarrow c = 0$, but $\begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$ unsolvable

$$\begin{pmatrix} 0 & 1 & 001 \\ A & B & C \\ 001 & 0 & 1 \end{pmatrix}$$

count balances of letters:

$$\begin{array}{rcl} 0: \; -a-b+2c &=& 0 \\ 1: \; -a+b &=& 0 \end{array}$$

$\Rightarrow a = b = c$, in any solution

for PCP(3), ratio $a : b : c$ is always determined

except for pathological cases (really?)

PCP Record Beavers

`http://www.informatik.uni-leipzig.de/~pcp/`

PCP(3,3)	$\begin{pmatrix} 0 & 1 & 001 \\ 001 & 0 & 1 \end{pmatrix}$	75	L,W
PCP(3,4)	$\begin{pmatrix} 101 & 1 & 010 \\ 1 & 01 & 1101 \end{pmatrix}$	216	Z
PCP(4,3)	$\begin{pmatrix} 000 & 0 & 11 & 10 \\ 0 & 111 & 0 & 100 \end{pmatrix}$	204	L
PCP(4,4)	$\begin{pmatrix} 1010 & 11 & 0 & 01 \\ 100 & 1011 & 1 & 0 \end{pmatrix}$	256	Z

R. J. Lorentz: Creating difficult instances of the PCP problem, CG 2000

Ling Zhao:
`http://www.cs.ualberta.ca/~zhao/PCP/list.htm`

Morphic PCPs

$$M(u,v) = \begin{pmatrix} 0 & 1 & u \\ v & 0 & 1 \end{pmatrix}, \text{ ex. } \begin{pmatrix} 0 & 1 & 001 \\ 001 & 0 & 1 \end{pmatrix}$$

starts $\begin{array}{l} 0\,0\,1 \\ 001\,001\,0 \end{array}$, ends $\begin{array}{r} 001\,001\,1\,1\,001 \\ 1\,1\,001 \end{array}$.

D0L words $0 \mapsto v, 1 \mapsto 0$ and $1 \mapsto \tilde{u}, 0 \mapsto 1$, at least initially. How do they meet?

$v = 01 \Rightarrow$ Fibonacci word $f = 01001010010\ldots$

Fibonacci PCP: $F(u) = M(u, 01)$ where $u \subset f$

$F(101) : 44, F(1001) : 78, F(001001) : 120,$
$F(10100101) : 216, F(0101001001) : 272$

— but $F(1001001)$ is unsolvable.

$F(1001001)$ *is unsolvable*

$$\left(\begin{array}{ccc} 0 & 1 & 1001001 \\ A & B & C \\ 01 & 0 & 1 \end{array}\right)$$

ends with C,
can never overtake,
starts with A (not C).

Lemma: we never reach 1100 or 0011
from the left, but we need this from the right.

Proof: take leftmost counterexample

case 1100: $\overbrace{1001001}^{C}\overset{B}{1} \Leftarrow \overset{B}{1}\overbrace{\underbrace{1}_{C}\underbrace{0}_{B}01001}^{C}$

case 0011:

(note: earlier C have both neighbours A)

$$\overbrace{1\underbrace{0}_{B}01\underbrace{0}_{B}01}^{C}\overset{B|C}{1} = \overbrace{\underbrace{1}_{A}\underbrace{0}_{B}\underbrace{01}_{A}\underbrace{0}_{B}\underbrace{01}_{A}}^{C}\underbrace{\overset{B|C}{1}}_{C}$$

$$\overbrace{1100}^{BBAA} \Leftarrow \overbrace{\underbrace{0}_{B}\underbrace{0}_{B}\underbrace{01}_{A}\underbrace{01}_{A}}^{AAABAB} \Leftarrow \overbrace{\underbrace{01}_{A}\underbrace{01}_{A}0}^{ABABA}\overbrace{\underbrace{1}_{A}\underbrace{0}_{B}\underbrace{01}_{A}\underbrace{0}_{B}01}^{C}$$

Growth

$\left(\begin{array}{cc} 0 & 10^n \\ A & B \\ 0^n1 & 0 \end{array}\right)$ has solution A^nB^n

because $0^n(10^n)^n = (0^n1)^n0^n$

shortest solution of PCP$(2, w)$ is $O(w)$?

$\left(\begin{array}{ccc} 0 & 0^n & 1^n0 \\ A & B & C \\ 00 & 1 & 0 \end{array}\right)$ has solution $A^{n^2}B^nC$

because $0^{n^2}0^{n^2}1^n0 = (00)^{n^2}1^n0$

shortest solution of PCP$(3, w)$ is $O(w^2)$?

Some Number Theory

$$\begin{pmatrix} 01 & 1^{2n}0 & 1 \\ A & B & C \\ 1 & 10 & 1010 \end{pmatrix}$$

example: for $n = 5$, shortest solution is

$CA^2C^2A^4C^4A^8C^8A^{16}C^6BCA^{14}C^4BCA^{10}B$

lower side ahead, difference $\in$ $(01)^k0$ or 1^k.

$(01)^{k-1}0 \xrightarrow{A^k} 1^{k-1}$ $\quad$ $1^{2n-1} \xrightarrow{B} \epsilon$

$1^{k-1} \xrightarrow{C^k} (01)^{2k-1}0 \xrightarrow{A^{2k}} 1^{2k-1}$

$1^{k-1} \xrightarrow{C^{k-2n}BC} (01)^{2(k-2n)+1}0 \xrightarrow{A^{2(k-2n)+2}} 1^{2(k-2n)+1}$

double or (subtract $2n - 1$ and double)

find least e such that $2^e \equiv 1 \bmod (2n - 1)$

extreme: $2n - 1$ is prime, 2 is primitive root.
but: length of solution is still $O(n^2)$.

Conjecture: PCP(3) is decidable

direction one: prove that

- morphic PCP(3) $\begin{pmatrix} 0 & 1 & u \\ v & 0 & 1 \end{pmatrix}$ is decidable
- reduce other PCP(3) to morphic case

  for example, find connection between
  PCP$(3,4)$ record $\begin{pmatrix} 101 & 1 & 010 \\ 1 & 01 & 1101 \end{pmatrix}$
  and Fibonacci $\begin{pmatrix} 0 & 1 & 10100101 \\ 01 & 0 & 1 \end{pmatrix}$
  both have shortest solution of $216 = 6^3$

direction two: prove that

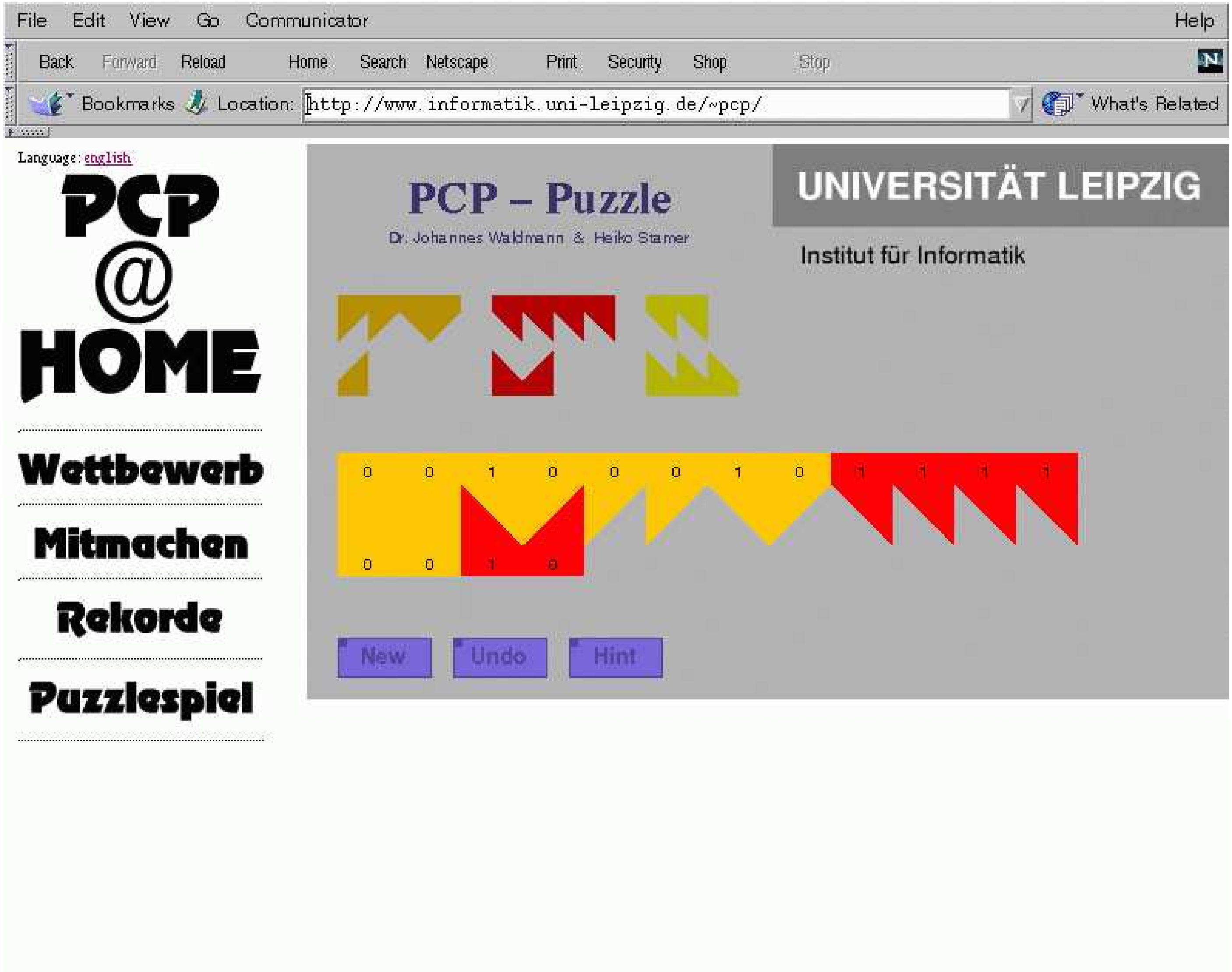
- shortest solution of PCP$(3, w)$ is $O(w^2)$

in any case: collect more data, that is,

- # find busier beavers

PCP@HOME

http://www.informatik.uni-leipzig.de/~pcp/



solve PCPs online!