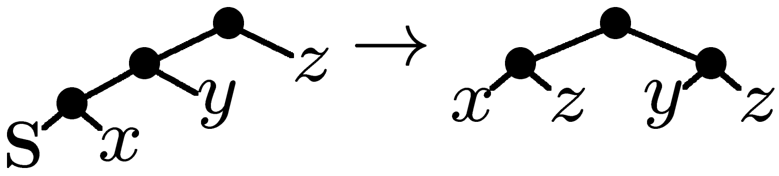
Deciding Head Normalization and Proving Top Termination for $\mathbf{CL}(\mathbf{S})$



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Motivation: Word Problem

input: X, Y *question*: does $X \leftrightarrow^* Y$ hold?

combinatory system
left linear, non overlapping, confluent
unique normal forms

if $\mathsf{nf}\ X = \mathsf{nf}\ Y$, then $X \leftrightarrow^* Y$.

non-termination!
want infinite normal forms (limits)
want top termination

look at head reductions
use regular tree expressions

will prove top termination
word problem will remain open
(need effective description of limits)

Top Termination: Definitions

redex positions in X: words in $\{L, R\}^*$

Def. A chain $X_1 \stackrel{p_1}{\to} X_2 \stackrel{p_2}{\to} \ldots$ is *top-terminating* iff $\exists\ m : \forall\ n > m : |p_n| > 0$.

Def. The TRS R is top-terminating iff all chains are.

Prop. If R is top-terminating, then all chains have a limit: $\forall\ d \geq 0 : \exists\ m : \forall\ m \geq n : |p_n| \geq d$

if R is orthogonal, then limits are unique.

Nachum Dershowitz, Stéfane Kaplan, and David A. Plaisted. Rewrite, rewrite, rewrite, rewrite, rewrite, ... *Theoretical Computer Science*, 83:71–96, 1991.

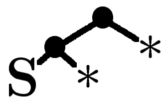
Richard Kennaway, Jan Willem Klop, Ronan Sleep, and Fer-Jan de Vries. Transfinite reductions in orthogonal term rewriting systems. *Information and Computation*, 119(1):18–38, May 1995.

Head reductions: Definitions

Def. $X \to_h Y$ iff $X \stackrel{p}{\to} Y$ with $p \in L^*$.

Thm. (Standardization) In $\mathbf{CL}(\ldots)$,
you can do all head reductions first.

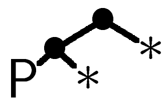
Prop. $\mathbf{CL}(\mathbf{S})$ is top-terminating
iff for each chain $X_1 \stackrel{p_1}{\to}_h X_2 \stackrel{p_2}{\to}_h \ldots$,
there is m such that $\forall\ n > m : |p_n| > 0$.

Def. (Head normal forms)
$\mathsf{H}_0 := \{X \mid \neg\exists Y : X \to_h Y\}$.
Prop. $\mathsf{H}_0 = \mathbf{S} \cup$  $\cup$ .

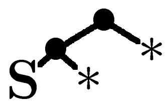
Def. (Head normalizing terms)
$\mathsf{H} := \{X \mid \exists Y \in \mathsf{H}_0 : X \to_h^* Y\}$.

Head reduction: Examples

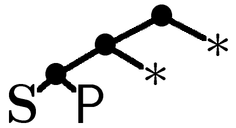
Def. P $=$ $\mathbf{S}$ $\cup$ 

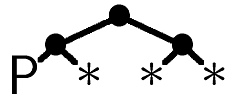
. **Prop.** 

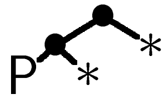
$\twoheadrightarrow$ H_0.

Proof. by induction. Base: 

$\subseteq$ H_0.

Step: 

$\to$ 

$\subseteq$ 

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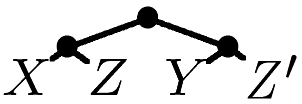
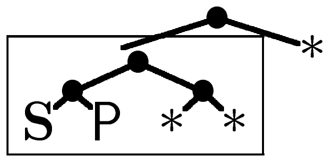
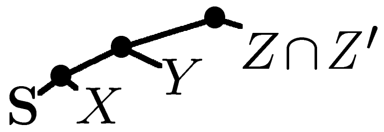
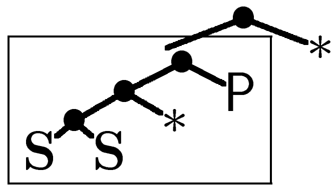
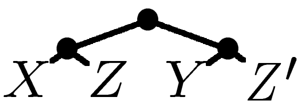
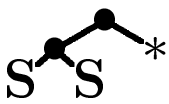
Find 

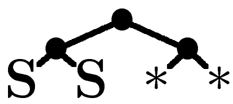
on left spine: 

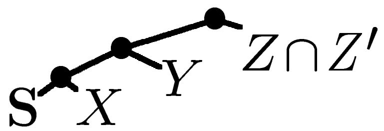
Replace by 

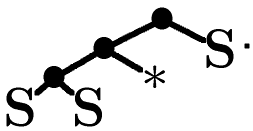
: 

Find 

in 

: 

Replace by 

: 

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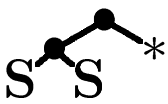
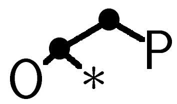
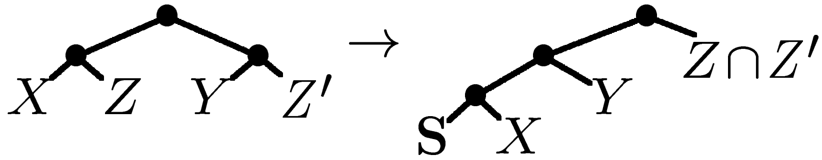
Def. 0 $=$ 

$\cup$ 

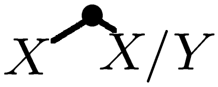
. **Prop.** 

$\twoheadrightarrow$ H_0.

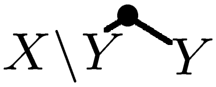
Inference of Predecessors

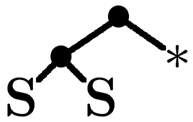
(B): 

Regular Expressions REX :

Def. $X/Y =$ 

$\cup\ Y$

Def. $X\backslash Y = X\ \cup$ 

Ex. P $= \mathbf{S}/\mathbf{S}$, O $=$ 

$\backslash\mathbf{S}$

extend $\{\mathbf{S},$  $\}$ by $\{*, \mathbf{S}/., .\backslash\mathbf{S}, \cap\}$

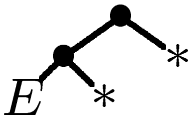
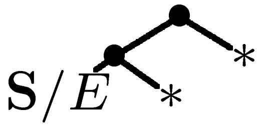
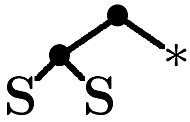
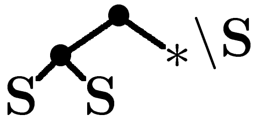
need operations on REX:

- introduce $\mathbf{S}/.$, $.\backslash\mathbf{S}$
- compute intersection
- simplify

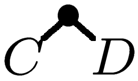
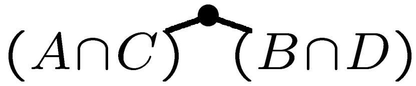
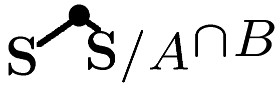
obtain head normalizing terms H
as B-closure of H_0

Computations on REX

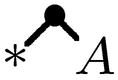
Introduce loops:

- replace  by 
- replace  by $0 =$ 

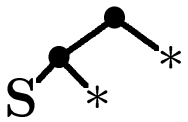
Compute intersections:

- replace  $\cap$  by 
- replace $(\mathbf{S}/A)\cap B$ by $A\cap B \ \cup$ 
- replace $(\mathbf{S}/A)\cap(\mathbf{S}/B)$
  by $\mathbf{S}/(A\cap B \ \cup \ A\cap(\mathbf{S}/B) \ \cup \ (\mathbf{S}/A)\cap B)$.

Simplify:

- replace  by $\mathbf{S}/A$
- replace  by 0
- replace  by $*$

Head reduction: Result

start with $\mathrm{H}_0 = \mathbf{S} \; \cup$  $\cup$ .

repeat infer-and-compute.

produces 81 expressions, then stops.

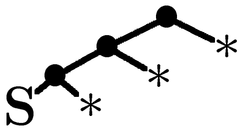
longest inference: 11 steps.

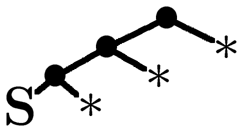
Thm. The set H of head normalizing $\mathbf{S}$ terms is regular.

Top Termination: Result

Assume $\mathbf{CL}(\mathbf{S})$ is not top-terminating.

Then there exists $X_1 \overset{p_1}{\to}_h X_2 \overset{p_2}{\to}_h \ldots$
such that $\{k \mid p_k = \epsilon\}$ is infinite.

If $p_k = \epsilon$, then $X_k \in$ .

Apply infer-and-compute to $\mathsf{T}_0 =$ .

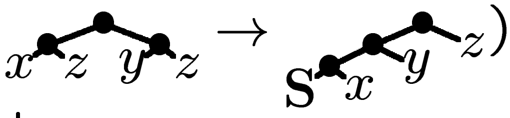
This returns a set $\mathsf{T} = \{\mathsf{T}_1, \ldots, \mathsf{T}_{1878}\} \subseteq \mathsf{REX}$.
(longest inference: 20 steps)

T is ordered, and each T_k is *local*:
$\forall\ X \in \mathsf{val}\ \mathsf{T}_k : \exists\ l < k : X \twoheadrightarrow \mathsf{val}\ \mathsf{T}_l$.

Thm. $\mathbf{CL}(\mathbf{S})$ is top-terminating.

Tree Automata with Small Loops

Def. Tree automaton (Σ, Q, Q_F, Δ)

completion of a tree automaton w.r.t. a rewrite system (here: 

)

is impossible in general:

- nonlinear lhs: need automata with checks
- need new states for right hand sides
- (nonlinear rhs: copying)

Def. *Small loop* automaton: $\exists$ partial order on Q with $\forall\ q \to f\,(q_1, \ldots, q_k) \in \Delta : \forall\ i : q \geq q_i$

interesting properties of SLA:

- l linear: terms encompassing l are SLA
- SLA is closed under union and intersection

not closed under complement:
$(ab)^+ = a\Sigma^*b \setminus (\Sigma^*aa\Sigma^* \cup \Sigma^*bb\Sigma^*)$

Summary

We proved for $\mathbf{CL}(\mathbf{S})$:

- Head normalization is decidable.
- The system is top-terminating.

We used tree automata methods:

- general: matching, intersection
- special: introduce/simplify loops

What remains open

- SLA (star free, dot depth?)
- connection between SLA and TRS
- word problem for $\mathbf{CL}(\mathbf{S})$
  (use classification T_k?)